JOURNAL OF DIFFERENTIAL EQUATIONS 71, 348–395 (1988)

Time-Decay of Scattering Solutions and Resolvent Estimates for Semiclassical Schrödinger Operators

XUE-PING WANG

Institut de Mathématiques et d'Informatique,
Université de Nantes, 44072 Nantes Cédex, France
and Institute of Mathematics, Peking University, 100871 Beijing, China

Received December 23, 1986; revised May 18, 1987

We study the semiclassical Schrödinger operator, $-h^2\Delta + V(x)$, $h \in]0, 1]$, and establish various results on the time-decay of scattering solutions and the semiclassical bounds for the powers of the resolvent. For short range potentials, the time-decay results obtained in this paper are the best possible and this enables us to conclude the equivalence between the non-trapping condition in classical mechanics and the uniform time-decay of wave functions in quantum mechanics.

1. INTRODUCTION

Let $H_0^h = -h^2\Delta$ and $H^h = H_0^h + V$ be Schrödinger operators on $\mathbf{R}^n$, where $h \in]0, 1]$ is a small parameter and V a smooth real potential on $\mathbf{R}^n$ satisfying for some $\varepsilon_0 > 0$

$$|\partial_x^\alpha V(x)| \leqslant C_\alpha \langle x \rangle^{-|\alpha| - \varepsilon_0}, \qquad x \in \mathbf{R}^n \tag{1.1}$$

for every $\alpha \in \mathbf{N}^n$. Here $\langle x \rangle \equiv (1 + |x|^2)^{1/2}$. In this paper, we are interested in the semiclassical propagation properties of scattering solutions of the Schrödinger equation

$$\begin{aligned} ih \frac{\partial}{\partial t} u(t, h) &= H^h u(t, h), \qquad && t \in \mathbf{R} \\ u(0, h) &= f(h), && f(h) \in L^2(\mathbf{R}^n), \end{aligned} \tag{1.2}$$

that is, solutions of (1.2) with initial data $f(h)$ supported in $\mathbf{R}_+$ in the spectral representation of H^h. We want to prove results on time-decay uniform with respect to $h \in]0, 1]$ for solutions of (1.2) in weighted-L^2 spaces. By Fourier–Laplace transform, these results are closely related to the smoothness of the resolvent for H^h. For fixed h, this kind of question was considered by a number of authors. See, for example, [13–15, 17, 21].

0022-0396/88 $3.00

Let us begin with giving a necessary condition to get the uniform time-decay for the solutions of problem (1.2). Let $U(t, h) = e^{-ih^{-1}tH^h}$ be the unitary group. Assume that for some $s > 0$ and $\varepsilon > 0$ we have

$$\|\langle x\rangle^{-s}\chi(H^h)\, U(t, h)\langle x\rangle^{-s}\|_{\mathscr{L}(L^2)} \leqslant C_{s,\varepsilon}(1 + |t|)^{-\varepsilon}, \qquad \forall t \in \mathbf{R},\, h \in]0, 1], \tag{1.3}$$

where $\chi \in C_0^\infty(]0, +\infty[)$. Let $\tilde{H}^h = -h\Delta + V(h^{1/2}x)$ and $\tilde{U}(t, h) = e^{-ith^{-1}\tilde{H}^h}$. Put

$$W_h(x_0, \xi_0) = \exp(ih^{-1}(x \cdot \xi_0 - x_0 \cdot D_x))$$

and

$$G(t,h) = W_h(x_0, \xi_0)^* \langle h^{1/2}x\rangle^{-s}\chi(\tilde{H}^h)\, \tilde{U}(t,h)\langle h^{1/2}x\rangle^{-s}\, \tilde{U}(t,h)^*\, W_h(x_0, \xi_0).$$

Since $\tilde{H}^h$ is unitarily equivalent to H^h, we have still

$$\|G(t, h)\|_{\mathscr{L}(L^2)} \leqslant C_{s,\varepsilon}(1 + |t|)^{-\varepsilon}, \qquad \text{for } t \in \mathbf{R} \text{ and } h \in]0, 1]. \tag{1.3)$'$}$$

Now by the results on semiclassical approximations (see [22, 25]), $G(t, h)$ converges strongly to $\langle x_0\rangle^{-s}\chi(|\xi_0|^2 + V(x_0))\langle x(-t; x_0, \xi_0)\rangle^{-s}$ as h tends to zero, where $(x(t), \xi(t))$ is the solution of the classical Hamiltonian system:

$$\dot{x}(t) = 2\xi(t), \qquad x(0) = y$$
$$\dot{\xi}(t) = -\nabla V(x(t)), \qquad \xi(0) = \eta.$$

In particular, it follows from (1.3)′ that

$$|x(t; x_0, \xi_0)|^s \geqslant C^{-1}|\chi(|\xi_0|^2 + V(x_0))|\langle x_0\rangle^{-s}(1 + |t|)^{\varepsilon}.$$

This means that in order to get a result of the form (1.3), it is necessary to assume that

$$\lim_{|t| \to +\infty} |x(t; x_0, \xi_0)| = +\infty \qquad \text{for} \quad (x_0, \xi_0) \in \mathbf{R}^{2n},$$

such that $\chi(|\xi_0|^2 + V(x_0)) \neq 0$. Therefore we are led to make the following condition on the solutions of classical Hamiltonian system. Let $p(x, \xi) = |\xi|^2 + V(x)$ and $J \subset]0, +\infty[$ be an open interval. We say that J is an interval of non-trapping energy iff

> for every subinterval $I \Subset J$ and for every $R > 0$, there is $t_0 > 0$ so that $|x(t; y, \eta)| > R$ for $|t| \geqslant t_0$ and $(y, \eta) \in p^{-1}(I)$ with $|y| < R$. (1.4)

The arguments given above show that for (1.3) to be true, it is necessary to choose $\chi \in C_0^\infty(]0, +\infty[)$ with support in some interval of non-trapping energy. In fact we will prove in this paper that for short range potentials this condition is also sufficient.

Notice that the condition (1.4) was used in [16, 19, 21]. In particular it is proved in [20] that under the conditions (1.1) and (1.4), for every $s > 1/2$, one has

$$\|\langle x\rangle^{-s} R(\lambda \pm i\varepsilon; h)\langle x\rangle^{-s}\| \leqslant Ch^{-1} \qquad \text{for} \quad \lambda \in I \Subset J, 0 < \varepsilon \leqslant 1$$

and for $h \in]0, 1]$. Here $\|\cdot\|$ is the norm of operators on $L^2(\mathbf{R}^n)$ and $R(z, h) = (H^h - z)^{-1}$. In this work, we will also give similar estimates for the powers of the resolvent.

Let $b_\pm \in C^\infty(\mathbf{R}^{2n})$ satisfying

$$|\partial_x^\alpha \partial_\xi^\beta b_\pm(x, \xi)| \leqslant C_{\alpha\beta}\langle x\rangle^{-|\alpha|} \qquad \text{on} \quad \mathbf{R}^{2n} \tag{1.5}$$

and that there exist $\sigma_\pm \in]-1, 1[$ such that

$$b_\pm(x, \xi) = 0, \qquad \text{if} \quad \pm\hat{x}\cdot\hat{\xi} \leqslant \pm\sigma_\pm, \tag{1.6}$$

where $\hat{x} = x/|x|$ and $\hat{\xi} = \xi/|\xi|$. Let $f \in \mathscr{S}(\mathbf{R}^n)$ and $\chi \in C_0^\infty(J)$, where J is an interval of non-trapping energy. Put

$$f(h) = \chi(H^h) f, \qquad f_\pm(h) = \chi(H^h)\, b_\pm(x, hD) f$$

with $b_\pm(x, hD)$ being the pseudo-differential operators associated to $b_\pm$ by the formula

$$a(x, hD)\, g(x) = (2\pi h)^{-n} \int\int e^{ih^{-1}(x-y)\cdot\xi} a(x, \xi)\, g(y)\, dy\, d\xi, \qquad g \in \mathscr{S}(\mathbf{R}^n).$$

Let $u(t, h)$ be the solutions of (1.2) with initial data $f(h)$ and $f_\pm(h)$, respectively. Then for every fixed t and h, $u(t, h)$ and $u_\pm(t, h)$ are in $\mathscr{S}(\mathbf{R}^n)$ (see [6, 22]). In this paper, we will prove the following results.

THEOREM 1. *Let V satisfy* (1.1) *with $\varepsilon_0 > 1$ (i.e., V is short range) and* (1.4). *Then*:

(i) *For every $s \geqslant 0$*

$$\|u(t, h)\|_{-s} \leqslant C_s(1 + |t|)^{-s}\|f\|_s, \qquad \text{for} \quad t \in \mathbf{R}, \forall h \in]0, 1].$$

(ii) *Assume that $b_\pm$ satisfies* (1.5) *and* (1.6). *Then for every $s, r \geqslant 0$,*

$$\|u_\pm(t, h)\|_{-r-s} \leqslant C_{r,s}(1 + |t|)^{-s}\|f\|_{-r}, \qquad \text{for} \quad \pm t > 0, \forall h \in]0, 1]$$

and

$$\|b_\mp(x, hD)\, u(t, h)\|_r \leqslant C_{r,s}(1 + |t|)^{-s}\|f\|_{r+s}, \qquad \text{for} \quad \pm t > 0, \forall h \in]0, 1].$$

(iii) *Assume that* $b_\pm$ *satisfies* (1.5) *and* (1.6) *with* $\sigma_- < \sigma_+$. *Then for every* $s, N \geqslant 0$,

$$\|b_\mp(x, hD)\, u_\pm(t, h)\|_s \leqslant C_{s,N}(1+|t|)^{-N}\|f\|_{-s}, \qquad \text{for} \quad \pm t > 0, \forall h \in]0,1].$$

Here $\|\cdot\|_s$ is the weighted L^2-norm: $\|f\|_s = \|\langle x\rangle^{-s} f\|_{L^2(\mathbf{R}^n)}$. Theorem 1 shows that for short range potentials, the uniform time-decay of scattering solutions is equivalent to the non-trapping condition (1.4). As a consequence of (i), Theorem 1, we get:

THEOREM 1′. *Suppose that* V *is a short range potential. Then the following three conditions are equivalent*:

(i) J *is an interval of non-trapping energy* (1.4);

(ii) *for some fixed* $s > 0$, $r > 0$ *and for every* $\chi \in C_0^\infty(J)$ *we have*

$$\|\langle x\rangle^{-s}\chi(H^h)\, U(t, h)\langle x\rangle^{-s}\| \leqslant C_\chi(1+|t|)^{-r}, \qquad t \in \mathbf{R}$$

uniformly in $h \in]0, 1]$;

(iii) *for every* $s > 0$, *and for every* $\chi \in C_0^\infty(J)$, *we have*:

$$\|\langle x\rangle^{-s}\chi(H^h)\, U(t, h)\langle x\rangle^{-s}\| \leqslant C_{\chi,s}(1+|t|)^{-s}, \qquad t \in \mathbf{R}$$

uniformly in $h \in]0, 1]$.

For long range potentials, we have not obtained such a good result and the decay of scattering solutions is derived from high order resolvent estimates.

THEOREM 2. *Let* V *be a long range potential satisfying* (1.1) (*with* $\varepsilon_0 > 0$) *and* (1.4). *Let* $I \Subset J$. *Then we have*:

(i) *For every* $m \in \mathbf{N}^*$ *and* $s > m - 1/2$

$$\|\langle x\rangle^{-s}(R(z, h))^m\langle x\rangle^{-s}\| \leqslant C_{m,s}h^{-m}, \quad \textit{for } h \in]0, 1];\ \mathrm{Re}\, z \in I,\ \mathrm{Im}\, z \neq 0.$$

(ii) *Let* $b_\pm$ *satisfy* (1.5) *and* (1.6). *Then for* $s > m - 1/2$,

$$\|\langle x\rangle^{s-m} b_\mp(x, hD)(R(z, h))^m\langle x\rangle^{-s}\| \leqslant C_{m,s}h^{-m},$$

and

$$\|\langle x\rangle^{-s}(R(z, h))^m b_\pm(x, hD)\langle x\rangle^{s-m}\| \leqslant C_{m,s}h^{-m}$$

for $h \in]0, 1]$; $\mathrm{Re}\, z \in I$ *and* $\pm\mathrm{Im}\, z > 0$.

(iii) *Let* $b_\pm$ *satisfy* (1.5) *and* (1.6) *with* $\sigma_- < \sigma_+$. *Then for any* $s > 0$,

$$\|\langle x\rangle^s\, b_\mp(x, hD)(R(z, h))^m\, b_\pm(x, hD)\langle x\rangle^s\| \leqslant C_{m,s}h^{-m}$$

for $h \in]0, 1]$; $\mathrm{Re}\, z \in I$, $\pm\mathrm{Im}\, z > 0$.

Recall that for fixed h, the smoothness of the boundary values $R(\lambda \pm i0; h)$ of the resolvent $R(z, h)$ was discussed in [13, 14, 28]. By a standard argument (see [14]), we obtain from (i) of Theorem 2 that for any $s > 0$ and $\varepsilon > 0$,

$$\|u(t, h)\|_{-s} \leqslant C_{s,\varepsilon} h^{-\varepsilon}(1 + |t|)^{-s+\varepsilon} \|f\|_s, \qquad t \in \mathbf{R}. \tag{1.7}$$

By the method used in the proof of Theorem 1, we can improve the decay rate in t.

THEOREM 3. *Let V be a long range potential satisfying* (1.1) *and* (1.4). *With the notations of Theorem* 1, *one has, for every $s, r \geqslant 0$ and $\varepsilon > 0$,*

$$\|u(t, h)\|_{-s} \leqslant C_{s,\varepsilon} h^{-\varepsilon}(1 + |t|)^{-s} \|f\|_s, \qquad \text{for } t \in \mathbf{R}, h \in]0, 1] \tag{1.8}$$

$$\|u_{\pm}(t, h)\|_{-r-s} \leqslant C_{r,s,\varepsilon} h^{-\varepsilon}(1 + |t|)^{-s} \|f\|_{-r}, \quad \text{for } \pm t > 0, h \in]0, 1] \tag{1.9}$$

$$\|b_{\mp}(x, hD)\, u(t, h)\|_r \leqslant C_{r,s,\varepsilon} h^{-\varepsilon}(1 + |t|)^{-s} \|f\|_{r+s}, \quad \text{for } \pm t > 0, h \in]0, 1]. \tag{1.10}$$

If $b_{\pm}$ satisfies (1.5) *and* (1.6) *with $\sigma_- < \sigma_+$, we have for every $N > 0$,*

$$\|b_{\mp}(x, hD)\, u_{\pm}(t, h)\|_s \leqslant C_{s,N,\varepsilon} h^{-\varepsilon}(1 + |t|)^{-N} \|f\|_{-s}, \text{ for } \pm t > 0, h \in]0, 1]. \tag{1.11}$$

Theorem 3 is not as beautiful as Theorem 1. But the decay rates in time in Theorem 3 are the best possible, as we can see in checking the solutions for the free Hamiltonian H_0^h. We believe that Theorem 1 is also true for long range potentials. For fixed h, the known results in the literature are usually of the form (1.7) [13–15]. However, in a paper to appear [29], Isozaki will prove a result similar to (1.8) for $h = 1$. Notice that for fixed h, one can drop the condition (1.4) and take $\chi \in C^\infty(\mathbf{R}^+)$ with bounded derivatives on $\mathbf{R}^+$ such that $\inf \operatorname{supp} \chi > 0$. See Theorem 5.4. The results proved in this paper can be applied in semiclassical scattering theory. See, for example, [23, 24] for $0 < h \leqslant 1$ and [27] for $h = 1$.

The proof of Theorems 1, 2, and 3 is based on the construction of an outgoing (resp. incoming) h-parametrix for (1.2) uniform with respect to $t > 0$ (resp. $t < 0$). In this connection, a key result is the existence of the phase functions which was established by Isozaki–Kitada [12]. Making use of these parametrices, we are able to prove first the desired results with $s = 1$ for Theorem 1 or $m = 1$ for Theorem 2. Notice that (i) of Theorem 2 with $m = 1$ is due to Robert–Tamura [20]. Then a suitable partition of unity enables us to obtain the general results by an inductive argument. Notice that for $|t|$ small, similar parametrices are powerful in the study of spectral properties for quantum Hamiltonians (see, for example, [8, 10] for $h=1$ and [5, 7] for $h \in]0, 1]$). But to study the propagation properties of scattering states, only parametrices global in t will be useful.

The organization of this paper is as follows. In Section 2, we construct an outgoing (resp. incoming) h-parametrix. For fixed h, the same construction was done by Isozaki–Kitada [12]. However, we cannot take $x \cdot \xi$ as phase function in the short range case and the transport equations we get are somewhat different from theirs. In Section 3 we establish some results for the free Hamiltonian H_0^h, which serve as models for general cases. In Section 4, we prove Theorem 1 for $0 \leqslant s \leqslant 1$. Notice that (i) was proved in [23] by commutator techniques (see also [24]). In Section 5 we prove Theorem 1 for $s \geqslant 0$. The group property of the solution $u(t, h)$ enables us to show Theorem 1 by a "partition of unity" of the form: $\chi(H^h) = b_+(x, hD; h) + b_-(x, hD; h) + R(h)$, where $b_\pm(h)$ is polynomial in h and satisfies (1.5) and (1.6), $R(h)$ is continuous from $L^{2,-N}$ to $L^{2,N}$ with $N>0$ large enough. In Section 6 we establish semiclassical microlocal resolvent estimates for H^h, that is to say, prove (ii), (iii) of Theorem 2 for $m=1$. For fixed h, these results were proved in [11] by stationary method. In Section 7 we prove Theorem 2 and in Section 8 we prove Theorem 3. In the Appendix, we collect some results on the symbolic calculus of a class of Fourier integral operators, which we need in this paper.

2. Construction of Outgoing and Incoming Parametrices

Assume that the potential V satisfies (1.1) for some $0 < \varepsilon_0 \leqslant 1$. Let $U(t, h)$ and $U_0(t, h)$ be the unitary groups defined by

$$U(t, h) = e^{-ith^{-1}tH^h} \qquad \text{and} \qquad U_0(t, h) = e^{-ih^{-1}tH_0^h}.$$

In this section we want to construct a h-parametrix for the problem (1.2) in the outgoing (resp. incoming) region uniformly with respect to $t>0$ (resp. $t<0$). This construction is equivalent to give a global approximation for $U(t, h)$ in the corresponding region. In fact, we will approximate $U(t, h)$ by $U_0(t, h)$ composed with some Fourier integral operators.

For $\sigma \in]0, 2[$ and $E>0$, set

$$\Omega_\pm(\sigma, E) = \{(x, \xi) \in \mathbf{R}^{2n};\ \pm\hat{x} \cdot \hat{\xi} > -1 + \sigma \quad \text{and} \quad |\xi| \geqslant E\}, \tag{2.1}$$

where $\hat{x} = x/|x|$ and $\hat{\xi} = \xi/|\xi|$. We call $\Omega_+(\sigma, E)$ (resp. $\Omega_-(\sigma, E)$) an outgoing (resp. incoming) region in phase space $\mathbf{R}^{2n}$ since for $\sigma \geqslant 1$, free particles situated in Ω_+ (resp. Ω_-) always move away from (resp. toward) the origin. We note also for $R > 0$:

$$\Omega_{\pm}(\sigma, E, R) = \{(x, \xi) \in \Omega_{\pm}(\sigma, E); |x| \geqslant R\} \tag{2.2}$$

and

$$\Omega(E) = \{(x, \xi) \in \mathbf{R}^{2n}; |\xi| \geqslant E\}.$$

The following result due to Isozaki–Kitada [11] is important in this work.

THEOREM 2.1. *Under the assumption* (1.1), *for every* $\varepsilon > 0$ *and* $E > 0$, *there exist two real functions* $\phi_{\pm} \in C^{\infty}(\mathbf{R}^{2n})$ *so that for* $R_1 = R_1(\varepsilon, E) > 1$ *large enough,* $\phi_{\pm}$ *solves the eikonal equation*

$$|\nabla_x \phi_{\pm}(x, \xi)|^2 + V(x) = |\xi|^2 \qquad \textit{in} \quad \Omega_{\pm}(\varepsilon, E, R_1) \tag{2.3}$$

and for any multiindices $\alpha, \beta \in \mathbf{N}^n$,

$$|\partial_x^\alpha \partial_\xi^\beta(\phi_{\pm}(x, \xi) - x \cdot \xi)| \leqslant C_{\alpha\beta} \langle x \rangle^{1 - \varepsilon_0 - |\alpha|} \langle \xi \rangle^{-1}$$

$$\left| \left(\frac{\partial^2}{\partial x_j \, \partial \xi_k} \phi_{\pm}(x, \xi) \right) - I \right| < 1/2$$

for $(x, \xi) \in \mathbf{R}^{2n}$. *Here we have put* $\langle x \rangle = (1 + |x|^2)^{1/2}$ *and* $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$.

In what follows, we will only construct an outgoing h-parametrix in the region $\Omega_+(\varepsilon, E, R)$. The incoming case can be treated similarly.

Let $\chi \in C^{\infty}(\mathbf{R}^n)$ so that $\chi(x) = 0$, if $|x| \leqslant 1/2$ and $\chi(x) = 1$, if $|x| \geqslant 1$. For $R > R_1$ define the function ϕ by

$$\phi(x, \xi) = (\phi_+(x, \xi) - x \cdot \xi) \, \chi(x/R) + x \cdot \xi.$$

Then ϕ solves the eikonal Eq. (2.3) in $\Omega_+(\varepsilon, E, R)$ and

$$|\partial_x^\alpha \partial_\xi^\beta(\phi(x, \xi) - x \cdot \xi)| \leqslant C_{\alpha\beta} R^{-\varepsilon_1} \langle x \rangle^{1 - \varepsilon_2 - |\alpha|} \qquad \text{on} \quad \mathbf{R}^n \times \mathbf{R}^n \tag{2.4}$$

for every $\varepsilon_1, \varepsilon_2 > 0$ so that $\varepsilon_1 + \varepsilon_2 = \varepsilon_0$. Here we have introduced an additional parameter R for later uses.

For $a \in C^{\infty}(\mathbf{R}^n \times \mathbf{R}^n)$ we have the equality

$$\begin{aligned} &e^{-ih^{-1}\phi}(-h^2 \Delta + V - |\xi|^2)(e^{ih^{-1}\phi} a) \\ &\quad = (|\nabla_x \phi|^2 + V - |\xi|^2) \, a - ih(2\nabla_x \phi \cdot \nabla_x a + (\Delta_x \phi) \, a) - h^2 \Delta_x a. \end{aligned} \tag{2.5}$$

Take a to be of the form: $a(h) = \sum_{h=0}^{N} h^j a_j$ with a_j to be determined. The right hand side of (2.5) can be written as

$$(|\nabla\phi|^2 + V - |\xi|^2)\, a(h) - ih(2\nabla\phi\, \nabla a_0 + (\Delta\phi)\, a_0)$$
$$- \sum_{j=1}^{N} h^{j+1}(2i\, \nabla\phi\, \nabla a_j + i(\Delta\phi)\, a_j + \Delta a_{j-1}) \tag{2.6}$$
$$- h^{N+2}\, \Delta a_N,$$

where the derivations are with respect to x. Equating the function before h^k, $1 \leqslant k \leqslant N+1$, to zero, we get the transport equations for a_k:

$$\begin{aligned} &2\nabla\phi \cdot \nabla a_0 + (\Delta\phi)\, a_0 = 0 \\ &2\nabla\phi \cdot \nabla a_j + (\Delta\phi)\, a_j = i\, \Delta a_{j-1}, \qquad j = 1, 2, ..., N. \end{aligned} \tag{2.7}$$

Remark that the transport equations used by Isozaki and Kitada are of the form

$$a_0 = 1 \qquad \text{on} \quad \mathbf{R}^n \times \mathbf{R}^n$$
$$2\nabla\phi \cdot \nabla a_j = -(\Delta\phi)\, a_{j-1} + i\, \Delta a_{j-1}, \qquad j = 1, 2, ..., N.$$

This is possible for fixed h, but impossible for constructing a h-parametrix. In order to solve the transport equations (2.7), we need some information on the characteristics of the vector field $\nabla\phi$. Let $\rho(t) = e^{t\nabla\phi}(x)$ be the solution of the Cauchy problem

$$\frac{\partial\rho}{\partial t}(t; x, \xi) = \nabla\phi(\rho(t, x, \xi), \xi), \qquad \rho(0; x, \xi) = x, \tag{2.8}$$

where ξ is regarded as a parameter. By (2.4), we can choose R so large that

$$|\partial_x \phi(x, \xi) - \xi| \leqslant \varepsilon E/4, \qquad (x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n. \tag{2.9}$$

LEMMA 2.2. *Under the condition* (2.9), *for* $|\xi| \geqslant E$, *the solution of* (2.8) *exists globally on* $\mathbf{R}^n$ *and there exists* $C > 0$ *so that*

$$C^{-1}|t| - |x| \leqslant |\rho(t; x, \xi)| \leqslant C(|x| + |t|\, |\xi|)$$

for $(x, \xi) \in \Omega(E)$. *In addition for fixed* ξ, *the mapping* $x \to p(t, x, \xi)$ *is a diffeomorphism on* $\mathbf{R}^n$ *for* $|t|$ *sufficiently small and we have the estimates,*

$$|\rho(t; x, \xi)| \geqslant C(|x| + t|\xi|), \tag{2.10}$$

for $t > 0$, $(x, \xi) \in \Omega_+(\varepsilon, E)$ *and*

$$|\partial_x^\alpha\, \partial_\xi^\beta(\rho(t; x, \xi) - x - \xi t)| \leqslant C_{\alpha\beta} \langle x \rangle^{-|\alpha| - \varepsilon_0}, \tag{2.11}$$

for $t > 0$, $(x, \xi) \in \Omega_+(\varepsilon, E)$ *and* $\alpha, \beta \in \mathbf{N}^n$ *with* $|\alpha| + |\beta| \geqslant 1$.

Proof. It is sufficient to write

$$\rho(t)=x+t\xi+\int_0^t \psi(\rho(\tau),\xi)\,d\tau, \tag{2.12}$$

where $\psi(x,\xi)=\nabla_x\phi(x,\xi)-\xi$. The Lemma follows from (2.4), (2.9) and the fact that for $(x,\xi)\in\Omega_+(\varepsilon,E)$ we have

$$|x+t\xi|\geqslant C(|x|+t|\xi|),\qquad \text{for}\quad t>0. \ \blacksquare$$

Since $\Delta\phi$ decreases as $0(|x|^{-1-\varepsilon_0})$ at infinity, by Lemma 2.2, $\Delta\phi(\rho(t))\in L^1(\mathbf{R})$ for every $(x,\xi)\in\Omega(E)$ and the function

$$F(t)=1/2\int_t^{+\infty}(\Delta\phi(\rho(\tau)))\,d\tau$$

is well defined and C^∞ on $\Omega(E)$. Put

$$a_0(x,\xi)=e^{F(0;x,\xi)}. \tag{2.13}$$

Then a_0 solves the first transport equation

$$2\nabla\phi\cdot\nabla a_0+(\Delta\phi)\,a_0=0\qquad \text{on}\quad \Omega(E)$$

and by (2.10) and (2.11),

$$|\partial_x^\alpha\,\partial_\xi^\beta(a_0(x,\xi)-1)|\leqslant C_{\alpha\beta}\langle x\rangle^{-|\alpha|-\varepsilon_0} \tag{2.14}$$

for $(x,\xi)\in\Omega_+(\varepsilon,E)$ and $\alpha,\beta\in\mathbf{N}^n$. By (2.4), we can easily prove the following.

PROPOSITION 2.3. *Let $1\leqslant k\leqslant N$. The functions a_k defined by induction*

$$a_k=a_0\int_0^{+\infty}\left\{-\frac{i}{2}\,\Delta a_{k-1}(\rho(\tau))\,e^{-F(\tau)}\right\}d\tau,\qquad k=1,2,...,N$$

with a_0 given by (2.13) *are C^∞ on $\Omega(E)$ and satisfy the transport equations*

$$2\nabla\phi\cdot\nabla a_k+(\Delta\phi)\,a_k=i\Delta a_{k-1},\qquad k=1,2,...,N \tag{2.7$_k$}$$

on $\Omega(E)$. We have the estimates

$$|\partial_x^\alpha\,\partial_\xi^\beta\,a_k(x,\xi)|\leqslant C_{\alpha\beta}\langle x\rangle^{-k-|\alpha|-\varepsilon_0},\qquad \textit{for}\quad (x,\xi)\in\Omega_+(\varepsilon,E)$$

for all $\alpha,\beta\in\mathbf{N}^n$.

Let χ be the function used in the definition of ϕ. Let ρ_1 and ρ_2 be C^∞ functions on $\mathbf{R}$ so that

$$\rho_1(t)=0, \quad \text{if} \quad t \leqslant -1+3\varepsilon/2 \quad \text{and} \quad \rho_1(t)=1, \quad \text{if} \quad t \geqslant -1+2\varepsilon$$

$$\rho_2(t)=0, \quad \text{if} \quad t \leqslant 3E/2 \quad \text{and} \quad \rho_2(t)=1, \quad \text{if} \quad t \geqslant 2E.$$

Put

$$a_N(x, \xi; h)=\left(\sum_{j=0}^{N} h^j a_j(x, \xi)\right) \chi(x/2R)\, \rho_1(\hat{x}\cdot\hat{\xi})\, \rho_2(|\xi|). \tag{2.15}$$

Then $a_N(h)$ is C^∞ function on $\mathbf{R}^{2n}$. On the support of $a_N(h)$, ϕ solves the eikonal Eq. (2.3)

$$(|\nabla\phi|^2+V-|\xi|^2)\, a_N(h)=0 \qquad \text{on} \quad \mathbf{R}^{2n}.$$

Let $p_N(h)$ be the function defined by

$$p_N(h)=h^{-1}e^{-ih^{-1}\phi}(H^h-|\xi|^2)(e^{ih^{-1}\phi}a_N(h)).$$

We denote $I(a, h)$ the Fourier integral operator with amplitude a and phase $\phi(x, \xi)$:

$$I(a, h)\, f(x)=(2\pi h)^{-n} \iint e^{ih^{-1}(\phi(x\cdot\xi)-\xi\cdot y)} a(x, \xi)\, f(y)\, dy\, d\xi. \tag{2.16}$$

For b an amplitude satisfying the estimates, for some $m_1, m_2 \in \mathbf{R}$,

$$|\partial_x^\alpha \partial_\xi^\beta b(x, \xi)| \leqslant C_{\alpha\beta}\langle x\rangle^{m_1-|\alpha|}\langle\xi\rangle^{m_2-|\beta|}$$

we define $U_N(t, h)$ by

$$U_N(t, h)=I(a_N(h), h)\, U_0(t, h)\, I(b, h)^*. \tag{2.17}$$

THEOREM 2.4. *Let $A_N(h)$ and $p_N(h)$ be defined by* (2.15) *and* (2.16). *We have*:

(i) *The amplitude $a_N(h)$ is C^∞ on $\mathbf{R}^{2n}$ and satisfies*

$$|\partial_x^\alpha \partial_\xi^\beta a_N(x, \xi; h)| \leqslant C_{\alpha\beta}\langle x\rangle^{-|\alpha|} \qquad (x, \xi)\in\mathbf{R}^{2n}$$

$$|\partial_x^\alpha \partial_\xi^\beta (a_N(x, \xi; h)-1)| \leqslant C_{\alpha\beta}\langle x\rangle^{-|\alpha|-\varepsilon_0} \qquad (x, \xi)\in\Omega_+(2\varepsilon, 2E; 2R)$$

uniformly in $h\in\,]0, 1]$ and $a_N(x, \xi; h)=0$ if $|x|\leqslant R$ or $\hat{x}\cdot\hat{\xi}\leqslant -1+\varepsilon$.

(ii) *For $p_N(h)$ we have the estimates*

$$|\partial_x^\alpha \partial_\xi^\beta p_N(x, \xi; h)| \leqslant C_{\alpha\beta}\langle x\rangle^{-1-|\alpha|} \qquad \text{on} \quad \mathbf{R}^{2n}$$

$$|\partial_x^\alpha \partial_\xi^\beta p_N(x, \xi; h)| \leqslant C_{\alpha\beta}h^{N+1}\langle x\rangle^{-N-|\alpha|} \qquad \text{on} \quad \Omega_+(2\varepsilon, 2E; 2R)$$

and

$$p_N(x, \xi; h) = 0 \quad \text{if} \quad |x| \leqslant R \quad \text{or} \quad |\xi| \leqslant R \quad \text{or} \quad \hat{x} \cdot \hat{\xi} \leqslant -1 + \varepsilon.$$

(iii) *The following formula is valid as continuous operators on* $\mathscr{S}(\mathbf{R}^n)$:

$$U(t, h)\, I(a_N(h), h)\, I(b, h)^* = U_N(t, h) + i \int_0^t U(t-s; h)\, R_N(s, h)\, ds, \qquad t \in \mathbf{R}, \tag{2.18}$$

where $R_N(x, h) = I(p_N(h), h)\, U_0(s, h)\, I(b, h)^*$.

Proof. (i) is clear by Proposition 2.3. To show (ii), we write down the expression for $p_N(h)$,

$$\begin{aligned} P_N(h) = & -2i(\nabla\phi \cdot \nabla\Psi)\, a_0 - \sum_{j=1}^{N} h^j (2i(\nabla\phi \cdot \nabla\Psi)\, a_j + (\Delta\Psi)\, a_{j-1} + 2\nabla\Psi \cdot \nabla a_{j-1}) \\ & - h^{N+1} \Delta(\Psi a_N), \end{aligned} \tag{2.19}$$

where $\Psi(x, \xi) = \chi(x/R)\, \rho_1(\hat{x} \cdot \hat{\xi})\, \rho_2(|\xi|)$, which satisfies the estimates

$$|\partial_x^\alpha \partial_\xi^\beta \Psi(x, \xi)| \leqslant C_{\alpha\beta} \langle x \rangle^{-|\alpha|} \langle \xi \rangle^{-|\beta|}.$$

Since Ψ is supported in $\Omega_+(\varepsilon, E, R)$, (ii) follows from Proposition 2.3

To prove (iii), recall first that $U(t, h)$ is a diffeomorphism on $\mathscr{S}(\mathbf{R})$ (see [6, 22]). Hence (2.18) is meaningful on $\mathscr{S}(\mathbf{R}^n)$. By the definition of $U_N(t, h)$, we have

$$ih \frac{\partial}{\partial t} U_N(t, h) = H^h U_N(t) + h R_N(t, h)$$

$$U_N(0, h) = I(a_N(h), h)\, I(b, h)^*$$

with $R_N(t, h) = I(p_N(h), h)\; U_0(t, h)\, I(b, h)^*$. By the Duhamel formula, we can express $U_N(t, h)$ by means of $U(t, h)$:

$$U_N(t, h) = U(t, h)\, I(a_N(h), h)\, I(b, h)^* - i \int_0^t U(t-s; h)\, R_N(s, h)\, ds.$$

This proves (iii). ∎

Notice that (2.18) is valid for $t \in \mathbf{R}$, but it is only useful for $t > 0$. Theorem 2.4 shows that $U_N(t, h)$ is a good approximation of $U(t, h)$ in the outgoing region $\Omega_+(2\varepsilon, 2E, 2R)$. It is clear that the utility of this parametrix depends on to what extent we can choose the initial data $I(a_N(h), h)\, I(b, h)^*$. At this point it is important to notice that $A_N(h)$ is elliptic; that is to say, the principal symbol a_0 is elliptic, on $\Omega_+(2\varepsilon, 2E, 2R)$. This enables us to prove that $I(a_N(h), h)\, I(b, h)^*$ can be taken as any pseudo-differential operator with symbol supported in an outgoing region.

Remark 2.5. Making use of the phase function ϕ_-, we can prove that for every N $N \geqslant 0$, there exist $a_{N,-}(h)$ and $p_{N,-}(h)$ satisfying:

$$\text{(i)} \qquad |\partial_x^\alpha \partial_\xi^\beta a_{N,-}(x, \xi; h)| \leqslant C_{\alpha\beta} < x >^{-|\alpha|} \qquad \text{on} \quad \mathbf{R}^{2n}$$

and $a_{N,-}(x, \xi; h) = 0$ if $|x| \leqslant R$, or $|\xi| \leqslant E$, or $\hat{x} \cdot \hat{\xi} \geqslant 1 - \varepsilon$.

$$\text{(ii)} \qquad |\partial_x^\alpha \partial_\xi^\beta p_{N,-}(x, \xi; h)| \leqslant C_{\alpha\beta} \langle x \rangle^{-1-|\alpha|} \qquad \text{on} \quad \mathbf{R}^{2n}$$

$$|\partial_x^\alpha \partial_\xi^\beta p_{N,-}(x, \xi; h)| \leqslant C_{\alpha\beta} h^{N+1} \langle x \rangle^{-N-|\alpha|}$$
$$\text{on } \Omega_-(2\varepsilon, 2E, 2R)$$

uniformly in $h \in]0, 1]$. In addition, we have the relation

$$\text{(iii)} \qquad U(t, h)\, I(a_{N,-}(h), h)\, I(b, h)^* = I(a_{N,-}(h), h)\, U_0(t, h)\, I(b, h)^* + i \int_0^t U(t-s, h)\, I(p_{N,-}(h), h)\, U_0(t, h)\, I(b, h)^*\, ds.$$

3. Some Results for the Free Hamiltonian

In this section, we consider the model of our problems: the free Hamiltonian H_0^h. Introduce first a particular class of symbols on $\mathbf{R}^{2n}$ (see [3] for more general cases).

Definition 3.1. For $m \in \mathbf{R}$, let S^m denote the class of symbols $a \in C^\infty(\mathbf{R}^{2n})$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leqslant C_{\alpha\beta} \langle x \rangle^{m-|\alpha|} \qquad \text{for} \quad (x, \xi) \in \mathbf{R}^{2n}.$$

We denote by $S_\pm$ the class of symbols $b_\pm \in S \equiv S^0$ such that there exist $r_1, r_2 > 0$ and $\sigma_\pm \in]-1, 1[$,

$$b_\pm(x, \xi) = 0, \qquad \text{if} \quad |x| \leqslant r_1 \text{ or } |\xi| \leqslant r_2$$
$$b_\pm(x, \xi) = 0, \qquad \text{if} \quad \pm \hat{x} \cdot \hat{\xi} < \pm \sigma_\pm.$$

Let ϕ be a real smooth function on $\mathbf{R}^{2n}$ satisfying for some $\varepsilon_0 > 0$

$$|\partial_x^\alpha \partial_\xi^\beta (\phi(x, \xi) - x \cdot \xi)| \leqslant C_{\alpha\beta} \langle x \rangle^{1-\varepsilon_0-|\alpha|} \langle \xi \rangle^{-1}$$

and

$$|(\partial_{x_j} \partial_{\xi_k} \phi(x, \xi)) - I| < 1/2 \qquad \text{on} \quad \mathbf{R}^{2n}.$$

We denote by $I(a, h)$ the Fourier integral operator with phase ϕ and amplitude a (see (2.16)).

PROPOSITION 3.2. *Let $b_{\pm} \in S_{\pm}$. Then for every $s \geqslant 0$ and $\rho \geqslant 0$ we have*

$$\|\langle x\rangle^s I(b_{\mp}, h)\, U_0(t, h)\langle x\rangle^{-s-\rho}\| \leqslant C(1+|t|)^{-\rho}, \qquad \text{for} \quad \pm t \geqslant 0 \tag{3.1}$$

and

$$\|\langle x\rangle^{-s-\rho} U_0(t, h)\, I(b_{\pm}, h)^* \langle x\rangle^s\| \leqslant C(1+|t|)^{-\rho}, \qquad \text{for} \quad \pm t \geqslant 0; \tag{3.2}$$

(3.1) *and* (3.2) *are uniform in* $h \in]0, 1]$.

Proof. It is sufficient to show (3.1). (3.2) follows by taking adjoint in (3.1). By the L^2-continuity of Fourier integral operators, (3.1) is evidently true for $s=\rho=0$. We only prove (3.1) for s and ρ integers. The general case follows by interpolations. For $f \in \mathscr{S}(\mathbf{R}^n)$ we can write

$$I(b_-, h)\, U_0(t, h)\, f(x) = (2\pi h)^{-n} \iint e^{ih^{-1}(\phi(x,\xi) - t|\xi|^2 - \xi \cdot y)} b_-(x, \xi)\, f(y)\, dy\, d\xi. \tag{3.3}$$

Put $S(x, y, \xi) = \phi(x, \xi) - x \cdot \xi - y \cdot \xi$. Notice that

$$|x - 2t\xi| \geqslant C(|x| + t|\xi|) \geqslant C'(|x| + t), \qquad C' > 0$$

for $(x, \xi) \in \operatorname{supp} b_-$ and $t > 0$. Making use of the operator $L_\xi = h\,|x - 2t\xi|^{-2}(x - 2t\xi) \cdot D_\xi$ which satisfies

$$L_\xi e^{ih^{-1}(x \cdot \xi - t|\xi|^2)} = e^{ih^{-1}(x \cdot \xi - t|\xi|^2)}$$

on the support of b_-, integrating (4.3) by part for $\rho + s$ times, we get

$$I(b_-, h)\, U_0(t, h)\, f(x) = (2\pi h)^{-n} \iint e^{ih^{-1}(x \cdot \xi - t|\xi|^2)} (L_\xi^*)^{\rho+s}$$
$$\times (e^{ih^{-1} S(x, y, \xi)} b_-(x, \xi))\, f(y)\, dy\, d\xi.$$

By an easily calculus, we get

$$(L_\xi^*)^{\rho+s}(e^{ih^{-1}S(x, y, \xi)} b_-(x, \xi)) = \sum_{|\alpha| \leqslant \rho+s} e^{ih^{-1}S(x, y, \xi)} h^{\rho+s-|\alpha|} b_\alpha(x, \xi, t)\, y^\alpha,$$

where $b_\alpha(x, \xi; t)$ satisfies

$$|\partial_x^\gamma \partial_\xi^\beta b_\alpha(x, \xi; t)| \leqslant C_{\gamma\beta}(\langle x\rangle + t)^{-\rho-s} \langle x\rangle^{(1-\varepsilon_0)(\rho+s-|\alpha|)-|\gamma|} \qquad \text{for} \quad t > 0. \tag{3.4}$$

Consequently, $I(b_-, h)\, U_0(t, h)$ may be written as

$$I(b_-, h)\, U_0(t, h) = \sum_{|\alpha| \leqslant \rho + s} h^{\rho + s - |\alpha|} I(b_\alpha(t), h)\, U_0(t, h)\, x^\alpha. \tag{3.5}$$

Notice that the support of $b_\alpha(t)$ is contained in $\operatorname{supp} b_-$ for all $t > 0$. If $|\alpha| = \rho + s$, applying Proposition A.2, we get

$$\|\langle x\rangle^s I(b_\alpha(t), h)\, U_0(t, h)\, x^\alpha f\| \leqslant C(1 + t)^{-\rho} \|\langle x\rangle^{\rho + s} f\|$$

uniformly in $t > 0$ and $h \in]0, 1]$. If $|\alpha| \leqslant \rho + s - 1$, we can apply L_ξ for $\rho + s - |\alpha|$ times and obtain an expression similar to (3.5),

$$I(b_\alpha(t), h)\, U_0(t, h) = \sum_{|\beta| \leqslant \rho + s - |\alpha|} h^{\rho + s - |\alpha| - |\beta|} I(b_{\alpha\beta}(t), h)\, U_0(t, h)\, x^\beta,$$

where

$$\begin{aligned} |\partial_x^\gamma \partial_\xi^\theta b_{\alpha\beta}(t)| &\leqslant C_{\gamma\theta}(\langle x\rangle + t)^{-2(\rho + s) + |\alpha|} \langle x\rangle^{(1 - \varepsilon_0)(2(\rho + s - |\alpha|) - |\beta|) - |\gamma|} \\ &\leqslant C'(\langle x\rangle + t)^{-(\rho + s)} \langle x\rangle^{(1 - 2\varepsilon_0)(\rho + s - |\alpha| - |\beta|) - |\gamma|} \end{aligned}$$

From (3.5) we conclude that $I(b_-, h)\, U_0(t, h)$ may be written as

$$I(b_-, h)\, U_0(t, h) = \sum_{|\alpha| \leqslant \rho + s} h^{k(\alpha)} I(d_\alpha(t), h)\, U_0(t, h)\, x^\alpha$$

with $d_\alpha(t)$ satisfying

$$|\partial_x^\gamma \partial_\xi^\theta d_\alpha(t)| \leqslant C_{\gamma\theta}(\langle x\rangle + t)^{-s-\rho} \langle x\rangle^{(1 - 2\varepsilon_0)(\rho + s - |\alpha|) - |\gamma|}$$

on $\mathbf{R}^{2n}$ uniformly in $t > 0$. Repeating this argument for a sufficient number of times, we deduce that

$$I(b_-, h)\, U_0(t, h) = \sum_{|\alpha| \leqslant \rho + s} h^{j(\alpha)} I(u_\alpha(t), h)\, U_0(t, h)\, x^\alpha, \tag{3.6}$$

where $j(\alpha) \geqslant 0$ and $u_\alpha(t)$ satisfies

$$|\partial_x^\gamma \partial_\xi^\beta u_\alpha(t)| \leqslant C_{\gamma\theta}(\langle x\rangle + t)^{-s-\rho} \langle x\rangle^{-|\gamma|}, \qquad \text{for} \quad t > 0.$$

By Proposition A.2 and (3.6), we get

$$\begin{aligned} \|\langle x\rangle^s I(b_-, h)\, U_0(t, h) \langle x\rangle^{-s-\rho}\| &\leqslant C \sum_\alpha \|\langle x\rangle^s I(u_\alpha(t), h)\| \\ &\leqslant C(1 + t)^{-\rho} \end{aligned}$$

uniformly in $t > 0$ and $h \in]0, 1]$. This proves Proposition 3.2. ∎

Put $R_0(z, h) = (H_0^h - z)^{-1}$ for $z \notin \mathbf{R}^+$. From Proposition 3.2, we get easily the following resolvent estimates.

COROLLARY 3.3. *Let $b_\pm \in S_\pm$. Then for every $s \geq 0$ and $\rho > 1$ we have*

$$\|\langle x\rangle^s I(b_\mp, h)\, R_0(\lambda \pm i0; h)\langle x\rangle^{-s-\rho}\| \leq Ch^{-1} \tag{3.7}$$

and

$$\|\langle x\rangle^{-s-\rho} R_0(\lambda \pm i0, h)\, I(b_\pm, h)^* \langle x\rangle^s\| \leq Ch^{-1} \tag{3.8}$$

uniformly in $\lambda \in \mathbf{R}_+$ and $h \in]0, 1]$.

Notice that (3.7) and (3.8) are not the best possible. In fact we have the following.

THEOREM 3.4. *Let $b_\pm \in S_\pm$ and $s > 1/2$. We have*

$$\|\langle x\rangle^{s-1} b_\mp(x, hD)\, R_0(\lambda \pm i0, h)\langle x\rangle^{-s}\| \leq Ch^{-1} \qquad (3.9)_\pm$$

$$\|\langle x\rangle^{-s} R_0(\lambda \pm i0, h)\, b_\pm(x, hD)\langle x\rangle^{s-1}\| \leq Ch^{-1}, \qquad h \in]0, 1]. \qquad (3.10)_\pm$$

These estimates are locally uniform in $\lambda > 0$.

Proof. We proof only $(3.9)_+$, because $(3.9)_-$ can be proven by the same method and (3.10) follows by taking the adjoint in (3.9). Observe first that if $b_-(x, \xi) = 0$ on a small neighborhood of $\{|\xi| = \sqrt{\lambda}\}$, $b_-(x, hD)\, R_0(\lambda + i0, h)$ is a pseudo-differential operator with symbol $b_-(x, \xi)(|\xi|^2 - \lambda)^{-1}$, which is uniformly bounded from $L^{2,s}$ to $L^{2,s}$, $\forall s \in \mathbf{R}$. Thus without loss of generality, we can assume that the support of b_- is contained in a sufficiently small neighborhood of $\{|\xi| = \sqrt{\lambda}\}$:

$$b_-(x, \xi) = 0, \qquad \text{if} \quad ||\xi|^2 - \lambda| > \varepsilon. \tag{3.11}$$

Take two partitions of unity on S^{n-1}: $\{\Psi_k\}$ and $\{\varphi_k\}$. By the condition on b_-, refining these partitions of unity if necessary, we have

$$b_-(x, \xi)\, \Psi_k(\hat{x})\, \varphi_j(\hat{\xi}) \equiv 0$$

if $\operatorname{supp} \Psi_k \cap \operatorname{supp} \varphi_j \neq \phi$. If $\operatorname{supp} \Psi_k \cap \operatorname{supp} \phi_j = \phi$, by a suitable rotation on S^{n-1}, we can assume that the support of $\Psi_k(\hat{x})$ is contained in $\{\hat{x}_n < 0\}$ and that of $\varphi_j(\hat{\xi})$ contained in $\{\hat{\xi}_n > 0\}$. Put

$$p(x, \xi) = b_-(x, \xi)\, \Psi_k(\hat{x})\, \varphi_j(\hat{\xi})$$

and

$$u(z, h) = R_0(z, h) f, \qquad f \in \mathcal{S}, \operatorname{Im} z > 0, \operatorname{Re} z = \lambda.$$

Notice that for $\varepsilon > 0$ sufficiently small (see (3.11)), the support of $p(x, .)$ as a function of ξ is strictly contained in the set

$$S = \{\xi = (\xi', \xi_n) \in \mathbf{R}^n; |\xi'| \in]0, \lambda^{1/2}[\text{ and } \xi_n > 0\}.$$

We can choose a real function $\chi \in C_0^\infty(\mathbf{R}^n)$, so that $\chi(\xi) = 1$ on $\operatorname{supp}_\xi p(x, .)$ and $\operatorname{supp} \chi \Subset S$. Then on the support of χ,

$$\operatorname{Re}(z - |\xi'|^2) \geqslant C\lambda > 0$$
$$\operatorname{Im}(z - |\xi'|^2) = \operatorname{Im} z \geqslant 0.$$

This shows that

$$Q(\xi, z) = (\xi_n + (z - |\xi'|^2)^{1/2})\, \chi(\xi)$$

is well defined and C^∞ on $\mathbf{R}^n$. By a simple calculus, we get

$$(hD_{x_n} - A(h\, D', z))\, \chi(hD)\, u(z, h) = Q(hD, z) f, \tag{3.12}$$

where

$$A(\xi', z) = \{z - |\xi'|^2\}^{1/2} \chi_1(\xi')$$

and $\chi_1 \in C_0^\infty(\mathbf{R}^{n-1})$ so that $\chi_1(\xi') \geqslant 0$ on $\mathbf{R}^{n-1}$, $\chi_1(\xi') = 1$ on $\operatorname{supp}_{\xi'} \chi(., \xi_n)$ and $\operatorname{supp} \chi_1 \Subset \{|\xi'| \subset]0, \lambda^{1/2}[\}$. It is clear that we have

$$\operatorname{Re} A(hD', z) \geqslant 0 \qquad \text{and} \qquad \operatorname{Im} A(hD', z) \geqslant 0 \tag{3.13}$$

in the sense of self-adjoint operator on $L^2(\mathbf{R}^n)$. Notice that for $\operatorname{Im} z > 0$, $u(z, h) \in \mathscr{S}(\mathbf{R}^n)$. Introduce the decomposition: $L^2(\mathbf{R}^n) = L^2(\mathbf{R}^{n-1}_{x'}))$. Put

$$v(x_n) = (\chi(hD)\, u(z, h))(., x_n)$$
$$g(x_n) = (Q(hD, z)\, f)(., x_n).$$

Then $v \in \mathscr{S}(\mathbf{R}, L^2(\mathbf{R}^{n-1}))$ and $g \in \mathscr{S}(\mathbf{R}; L^2(\mathbf{R}^{n-1}))$. From (3.12) and (3.13), we get

$$\begin{aligned}\frac{d}{dx_n} \|v(x_n)\|^2_{L^2(\mathbf{R}^{n-1})} = h^{-1}\{&-\langle \operatorname{Im} A(hD', z)\, v(x_n), v(x_n)\rangle_{L^2(\mathbf{R}^{n}-1)} \\ &+ 2 \operatorname{Re}\langle v(x_n), g(x_n)\rangle_{L^2(\mathbf{R}^{n-1})}\} \\ \leqslant{}& 2h^{-1} \|v(x_n)\|_{L^2(\mathbf{R}^{n-1})}\, \|g(x_n)\|_{L^2(\mathbf{R}^{n-1})}.\end{aligned}$$

It follows easily that

$$\|v(0)\|^2_{L^2(\mathbf{R}^{n-1})} \leqslant 2h^{-1} \int_{-\infty}^0 \|v(x_n)\|\, \|g(x_n)\|\, dx_n$$

and for $s > 1/2$,

$$\begin{aligned}
\int_{-\infty}^{0} & (1 - x_n)^{2(s-1)} \|v(x_n)\|^2 dx_n \\
& \leqslant Ch^{-1} \int_{-\infty}^{0} (1 - x_n)^{2s-1} \|v(x_n)\| \, \|g(x_n)\| \, dx_n \\
& \leqslant C \left\{ h^{-2} \varepsilon^{-1} \int_{-\infty}^{0} (1 - x_n)^{2s} \|g(x_n)\|^2 \, dx_n \right. \\
& \qquad \left. + \varepsilon \int_{-\infty}^{0} (1 - x_n)^{2(s-1)} \|v(x_n)\|^2 \, dx_n \right\}
\end{aligned}$$

for every $\varepsilon > 0$. For $\varepsilon > 0$ sufficiently small, we get

$$\begin{aligned}
\int_{-\infty}^{0} (1 - x_n)^{2(s-1)} \|v(x_n)\|^2 dx_n & \leqslant Ch^{-2} \int_{-\infty}^{0} (1 - x_n)^{2s} \|g(x_n)\|^2 \, dx_n \\
& \leqslant Ch^{-2} \|Q(hD, z) f\|^2_{L^{2,s}(\mathbf{R}^n)} \qquad (3.14)
\end{aligned}$$

with the constant $C > 0$ independent of $h \in]0, 1]$ and $\operatorname{Im} z \in]0, 1]$. Let $\varphi \in C^\infty(S^{n-1})$ so that $\varphi(\hat{x}) = 1$ on $\operatorname{supp} \Psi_k$ and $\operatorname{supp} \varphi \subset \{\hat{x} \in S^{n-1}, \hat{x}_n < 0\}$. Let $\rho \in C^\infty(\mathbf{R}^n)$ so that $\rho(x) = 0$ if $|x| \leqslant 1/2$ and $\rho(x) = 1$ if $|x| \geqslant 1$. Put

$$\chi_2(x) = \rho(x) \, \varphi(\hat{x}).$$

Then $\|\chi_2(., x_n) \, v(x_n)\|_{L^2(\mathbf{R}^{n-1})} \leqslant C \|v(x_n)\|_{L^2(\mathbf{R}^{n-1})}$ and on the support of χ_2, there exists $C > 0$ so that

$$C^{-1} \langle x \rangle \leqslant (1 - x_n) \leqslant C \langle x \rangle.$$

From (3.14) it follows that

$$\|\chi_2(x) \, \chi(hD) \, u(z, h)\|^2_{L^{2,s-1}(\mathbf{R}^n)} \leqslant Ch^{-2} \|f\|^2_{L^{2,s}}. \qquad (3.15)$$

Since $\chi_2(x) \, \chi(\xi) = 1$ on the support of p, by the results on symbolic calculus for pseudo-differential operators, we have

$$p(x, hD) \, \chi_2(x) \, \chi_1(hD) = p(x, hD) + h^N R_N(x, hD, h) \qquad \text{for every } N \geqslant 1, \qquad (3.16)$$

where $R_N(x, \xi; h)$ satisfies the estimates

$$|\partial_x^\alpha \partial_\xi^\beta R_N(x, \xi; h)| \leqslant C_{\alpha\beta} \langle x \rangle^{-N-|\alpha|} \langle \xi \rangle^{-N-|\beta|}, \qquad (x, \xi) \in \mathbf{R}^{2n}$$

uniformly in $h \in]0, 1]$. Since $p(x, hD)$ is continuous from $L^{2,s-1}$ to $L^{2,s-1}$, we deduce from (3.15) and (3.16) that

$$\|p(x, hD)\, u(z, h)\|_{L^{2,s-1}} \leqslant Ch^{-1}\|f\|_{L^{2,s}} + h^N\|u(z, h)\|_{L^{2,N}}$$
$$\leqslant Ch^{-1}\|f\|_{L^{2,s}}, \qquad h \in]0, 1].$$

Pasting together the non-zero terms appeared in the partition of unity at the beginning we get

$$\|b_-(x, hD)\, R_0(z, h)\, f\|_{L^{2,s-1}} \leqslant Ch^{-1}\|f\|_{L^{2,s}}, \qquad h \in]0, 1]$$

for $\operatorname{Re} z = \lambda$ and $\operatorname{Im} z \in]0, 1]$. This proves $(3.9)_+$. ∎

If we make a bilateral localisation, we can get better results.

PROPOSITION 3.5. *Let $b_\pm \in S_\pm$ such that there exists $\sigma_\pm \in]-1, 1[$ with $\sigma_- < \sigma_+$ and*

$$b_+(x, \xi) = 0 \qquad \text{if} \quad \hat{x} \cdot \hat{\xi} < \sigma_+$$
$$b_-(x, \xi) = 0 \qquad \text{if} \quad \hat{x} \cdot \hat{\xi} > \sigma_-.$$

Then for every $s \geqslant 0$ and $N \geqslant 0$ we have

$$\|\langle x\rangle^s I(b_\mp, h)\, U_0(t, h)\, I(b_\pm, h)^* \langle x\rangle^s\| \leqslant C_N(1 + |t|)^{-N}, \qquad \pm t > 0 \tag{$3.17_\pm$}$$

uniformly in $h \in]0, 1]$.

Proof. For $f \in \mathscr{S}(\mathbf{R}^n)$, we can write $I(b_-, h)\, U_0(t, h)\, I(b_+, h)^* f$ as

$$I(b_-, h)\, U_0(t, h)\, I(b_+, h)^* f(x)$$
$$= (2\pi h)^{-n} \int\int e^{ih^{-1}(\phi(x,\xi) - \phi(y,\xi) - t|\xi|^2)} b_-(x, \xi)\, \overline{b_+(y, \xi)}\, f(y)\, dy\, d\xi. \tag{3.18}$$

Put

$$S(x, y, \xi) = \phi(x, \xi) - x \cdot \xi - \phi(y, \xi) + y \cdot \xi$$
$$a(x, y, \xi; h) = e^{ih^{-1}S(x, y, \xi)} b_-(x, \xi)\, \overline{b_+(y, \xi)}.$$

Notice that if $\sigma_+ > 0$, we hae for $t \geqslant 0$ and $(y, \xi) \in \operatorname{supp} b_+$

$$(y + 2t\xi) \cdot \xi \geqslant \sigma_+ |y|\, |\xi| - 2t|\xi|^2 \geqslant \sigma_+ |y + 2t\xi|\, |\xi|.$$

If $\sigma_+ \leqslant 0$, $\sigma_- < \sigma_+ \leqslant 0$. Then for $t \geqslant 0$ and $(x, \xi) \in \operatorname{supp} b_-$, one has

$$(x - 2t\xi) \cdot \xi \leqslant \sigma_- |x|\, |\xi| - 2t|\xi|^2 \leqslant \sigma_- |x + 2t\xi|\, |\xi|.$$

But if $\hat{x} \cdot \hat{\xi} \leqslant \sigma_-$ and $\hat{y} \cdot \hat{\xi} \geqslant \sigma_+$ with $\sigma_- < \sigma_+$, one has always

$$|x - y| \geqslant C(|x| + |y|)$$

for some $C = C(\sigma_+, \sigma_-) > 0$ independent of $\hat{\xi} \in S^{n-1}$. Consequently on the support of $a(h)$

$$\begin{aligned} |x - (y + 2t\xi)| &\geqslant C \max(|x - 2t\xi| + |y|, |x| + |y + 2t\xi|) \\ &\geqslant C'(|x| + |y| + t|\xi|), \qquad t \geqslant 0. \end{aligned} \tag{3.19}$$

Define the operator L_ξ by

$$L_\xi = h|x - y - 2t\xi|^{-2}(x - y - 2t\xi) \cdot D_\xi.$$

Then $L_\xi e^{ih^{-1}(x \cdot \xi - t|\xi|^2 - y \cdot \xi)} = e^{ih^{-1}(x \cdot \xi - t|\xi|^2 - y \cdot \xi)}$. From (3.18) we get for every $N \geqslant 0$,

$$\begin{aligned} &I(b_-, h)\, U_0(t, h)\, I(b_+, h)^* f(x) \\ &\quad = (2\pi h)^{-n} \iint e^{ih^{-1}(x \cdot \xi - t|\xi|^2 - y \cdot \xi)} (L_\xi^*)^N a(h)\, f(y)\, dy\, d\xi. \end{aligned} \tag{3.20}$$

Applying (3.19) we get an expression for $(L_\xi^*)^N a(h)$,

$$(L_\xi^*)^N a(h) = e^{ih^{-1}S} \sum_{j=0}^{N} h^j C_j(t),$$

with $C_j(t)$ satisfying

$$\begin{aligned} |\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma C_j(x, y, \xi; t)| &\leqslant C_{\alpha\beta\gamma}(\langle x \rangle + \langle y \rangle + t)^{-N} \langle x \rangle^{N(1-\varepsilon_0) - |\alpha|} \\ &\quad \times \langle y \rangle^{N(1-\varepsilon_0) - |\beta|} \\ &\leqslant C'_{\alpha\beta\gamma}(1 + t)^{-\rho N} \langle x \rangle^{-\rho N - |\alpha|} \langle y \rangle^{-\rho N - |\beta|}, \qquad t \geqslant 0, \end{aligned} \tag{3.21}$$

where we have set $\rho = \varepsilon_0/3$. For given s, M, take N large enough. We deduce from (3.20), (3.21), and Proposition A.2 that

$$\|\langle x \rangle^s I(b_-, h)\, U_0(t, h)\, I(b_+, h)^* \langle x \rangle^s\| \leqslant C_{s,M}(1 + t)^{-M} \qquad \text{for} \quad t \geqslant 0$$

uniformly in $h \in\,]0, 1]$. This proves $(3.17)_+$; $(3.17)_-$ follows by taking the adjoint of $(3.17)_+$. ∎

Proposition 3.5 gives immediately the following estimate on the resolvent $R_0(z, h)$.

COROLLARY 3.6. *Under the assumptions of Proposition* 3.5, *for every* $s \geqslant 0$, *one has*

$$\|\langle x\rangle^s I(b_\mp, h)\, R_0(\lambda \pm i0; h)\, I(b_\pm, h)^* \langle x\rangle^s\| \leqslant C_2 h^{-1}, \qquad h \in]0, 1]$$

uniformly in $\lambda \in \mathbf{R}_+$.

4. MICRO-LOCAL ENERGY DECAY FOR SHORT RANGE POTENTIALS

In this section the assumptions on the potential V are the following:

V is short range (i.e., verifying (1.1) for some $\varepsilon_0 > 1$) and satisfies the condition (1.4). (4.1)

Surely V can be regarded as a long range potential and the construction of outgoing and incoming parameters given in Section 2 remains valid with $\varepsilon_0 = 1$. In this section we will prove micro-local energy decay for scattering solutions. These results are crucial for the proof of Theorem 1. Recall first the following local energy decay proved in [23] (see also [24]). For the reader's sake, we will give a sketch of the proof.

THEOREM 4.1. *Under the assumptions* (4.1), *let* $\chi \in C_0^\infty(J)$. *Then for* $s \in [0, 1]$ *we have*

$$\|\langle x\rangle^{-s}\chi(H^h)\, U(t, h)\langle x\rangle^{-s}\| \leqslant C(1 + |t|)^{-s} \qquad \textit{for} \quad t \in \mathbf{R} \tag{4.2}$$

uniformly in $h \in]0, 1]$.

Proof. Let $A(h) = h(x \cdot \nabla_x + \nabla_x \cdot x)/2i$. We can verify the following commutator relation on $\mathscr{S}(\mathbf{R}^n)$:

$$A(h)\, U(t, h) = U(t, h)\, A(h) + 2tH^h U(t, h) + \int_0^t U(s - t, h)\, \tilde{V} \cdot U(-s, h)\, ds \tag{4.3}$$

with $\tilde{V} = x \cdot \nabla V - 2V$ satisfying still (1.1). Recall that under the assumptions (1.1) (for $\varepsilon_0 > 0$) and (1.4), it is proved in [20] that

$$\|\langle x\rangle^{-s} R(\lambda \pm i0; h)\langle x\rangle^{-s}\| \leqslant Ch^{-1} \tag{4.4}$$

uniformly with respect to $h \in]0, 1]$ and $\lambda \in I \Subset J$. Suppose now that V is of short range. From (4.1), (4.4) and the results on H^h smoothness [18], we deduce from (4.3) that

$$\|(A(h) + i)^{-1}\chi(H^h)\, U(t, h)(A(h) + i)^{-1}\| \leqslant C(1 + |t|)^{-1}. \tag{4.5}$$

Take $f \in C_0^\infty(\mathbf{R}_+)$ so that $f\chi = \chi$. Since $\langle x \rangle^{-1} f(H^h)(A(h)+i)$ extends to a bounded operator on $L^2(\mathbf{R}^n)$, we deduce from (4.5) that

$$\| \langle x \rangle^{-1} \chi(H^h)\, U(t,h) \langle x \rangle^{-1} \| \leqslant C(1+|t|)^{-1}, \qquad t \in \mathbf{R} \tag{4.6}$$

uniformly in $h \in]0, 1]$. This proves (4.2) for $s = 1$. The general case follows by interpolation. ∎

For fixed h, by the same method as the proof of Theorem 4.1, we can show that under the condition (1.1) with $\varepsilon_0 > 1$, for $\chi \in C^\infty(\mathbf{R}_+)$ with $\inf \operatorname{supp} \chi > 0$, for every $s \in [0, 1]$, we have

$$\| \langle x \rangle^{-s} \chi(H)\, U(t)\, \langle x \rangle^{-s} \| \leqslant C(1+|t|)^{-s} \qquad \text{for} \quad t \in \mathbf{R}. \tag{4.7}$$

Here $U(t)$ is the unitary group associated with $H = -\Delta + V$.

One of the main results in this section is the following time-decay of scattering solutions in one-sided micro-local form.

THEOREM 4.2. *Under the assumptions* (4.1), *let* $\chi \in C_0^\infty(J)$ *and* $b_\pm \in S_\pm$. *Then we have, for every* $s \geqslant 0$ *and* $r \in [0, 1]$,

$$\| \langle x \rangle^{s} b_\pm(x, hD)\, U(t,h)\, \chi(H^h) \langle x \rangle^{-s-r} \| \leqslant C(1+|t|)^{-r}, \qquad \textit{for} \quad \mp t > 0 \tag*{$(4.8)_\pm$}$$

and

$$\| \langle x \rangle^{-s-r} \chi(H^h)\, U(t,h)\, b_\pm(x, hD) \langle x \rangle^{s} \| \leqslant C(1+|t|)^{-r}, \qquad \textit{for} \quad \pm t > 0, \tag*{$(4.9)_\pm$}$$

where (4.8) *and* (4.9) *are uniform with respect to* $h \in]0, 1]$.

Recall that for every $\varepsilon > 0$, $E > 0$, we have constructed an outgoing h-parametrix in the region $\Omega_+(\varepsilon, E, R)$, such that

$$U(t,h)\, I(a_N(h), h)\, I(b,h)^* = U_N(t,h) + i \int_0^t U(t-s, h)\, R_N(s,h)\, ds, \qquad t \in \mathbf{R}, \tag{4.10}$$

where

$$U_N(t,h) = I(a_N(h), h)\, U_0(t,h)\, I(b,h)^*$$
$$R_N(r,h) = I(p_N(h), h)\, U_0(r,h)\, I(b,h)^*$$

with the amplitudes $a_N(h)$ and $p_N(h)$ satisfying (i) and (ii) of Theorem 2.4, $b \in S^m$ being arbitrary.

Take $b \in S_+$. Then there exists $\sigma \in]-1, 1[$ such that

$$b(x, \xi) = 0, \qquad \text{if} \quad \hat{x} \cdot \hat{\xi} < \sigma.$$

For $\varepsilon, E > 0$ suficiently small, the support of b is contained in $\Omega_+(2\varepsilon, 2E)$. Choose a function $\rho \in C_0^\infty(\mathbf{R})$ such that

$$\begin{aligned} \rho(r) &= 1 \qquad \text{if} \quad r \geqslant -1 + 2\varepsilon; \\ &= 0 \qquad \text{if} \quad r \leqslant -1 + \varepsilon. \end{aligned}$$

Set $p_{N,1}(x, \xi; h) = \rho(\hat{x}.\hat{\xi})\, p_N(x, \xi; h)$ and $p_{N,2}(x, \xi; h) = p_N(x, \xi; h) - p_{N,1}(x, \xi; h)$. According to Theorem 2.4,

$$|\partial_x^\alpha \partial_\xi^\beta p_{N,1}(x, \xi; h)| \leqslant C_{\alpha\beta} h^{N+1} \langle x \rangle^{-N-|\alpha|-1} \tag{4.11}$$

and $p_{N,2}(h) \in S_-$ is of disjoint support with b. For $s \geqslant 0$ so that $2s + 3 \leqslant N$, we get from Proposition 3.2 that

$$\|\langle x \rangle^{s+1} I(p_{N,1}(h), h)\, U_0(r, h)\, I(b, h)^* \langle x \rangle^s\| \leqslant C h^{N+1} (1+r)^{-2}, \qquad r \geqslant 0$$

for $I(p_{N,1}(h), h)$ is uniformly continuous from $L^{2,-s-2}$ to $L^{2,s+1}$ by (4.11) and Proposition A.2. Applying Proposition 3.5, we get

$$\|\langle x \rangle^{s+1} I(p_{N,2}(h), h)\, U_0(r, h)\, I(b, h)^* \langle x \rangle^s\| \leqslant C (1+r)^{-2}, \qquad r \geqslant 0$$

uniformly in $h \in]0, 1]$. From Theorem 4.1, it follows that

$$\begin{aligned} &\left\| \langle x \rangle^{-s-1} \int_0^t \chi(H^h)\, U(t-r, h)\, R_N(r, h) \langle x \rangle^s \, dr \right\| \\ &\quad \leqslant C \int_0^t \|\langle x \rangle^{-1} \chi(H^h)\, U(t-r, h) \langle x \rangle^{-1}\| (1+r)^{-2} \, dr \\ &\quad \leqslant C' \int_0^t (1+t+r)^{-1} (1+r)^{-2} \, dr \leqslant C''(1+t)^{-1} \qquad \text{for} \quad t \geqslant 0 \end{aligned}$$

uniformly in $h \in]0, 1]$. By Proposition A.2, $\chi(H^h)\, I(a_N(h), h)$ is uniformly continuous from $L^{2,-1-s}$ to $L^{2,-1-s}$. Consequently,

$$\|\langle x \rangle^{-s-1} \chi(H^h)\, I(a_N(h), h)\, U_0(t, h)\, I(b, h)^* \langle x \rangle^s \leqslant C(1+t)^{-1} \quad \text{for } t \geqslant 0.$$

By (4.10) we have proved the following:

LEMMA 4.3. *For $b \in S_+$, take ε and $E > 0$ sufficiently small. Then for every $s \geqslant 0$ and $N \geqslant 2s + 3$, we have*

$$\|\langle x \rangle^{-s-1} \chi(H^h)\, U(t, h)\, I(a_N(h), h)\, I(b, h)^* \langle x \rangle^s\| \leqslant C(1+t)^{-1}, \qquad t \geqslant 0$$

uniformly in $h \in]0, 1]$.

Notice that the phase function ϕ depends on the parameter $R>0$ and satisfies

$$|\partial_x^\alpha \partial_\xi^\beta(\phi(x,\xi)-x\cdot\xi)| \leqslant C_{\alpha\beta} R^{-\varepsilon_1}\langle x\rangle^{1-\varepsilon_2-|\alpha|}\langle \varepsilon\rangle^{-1} \tag{4.12}$$

(see (2.4)) with $\varepsilon_1, \varepsilon_2>0$ and $\varepsilon_1+\varepsilon_2=\varepsilon_0$ (here $\varepsilon_0=1$). Relation (4.12) is uniform in $R \gg 1$. For R large enough, the mappings $x \to \nabla_\xi\phi(x,\xi)$ and $\xi\to\nabla_x\phi(x,\xi)$ are global diffeomorphism on $\mathbf{R}_x^n$ and $\mathbf{R}_\xi^n$, respectively. Let $x\to\eta(x,\xi)$ be the inverse diffeomorphism for the mapping $\xi\to\nabla_x\phi(x,\xi)$.

By the definition of $S_\pm$, for $b\in S_+$, there exist positive numbers σ, r_1, r_2 such that the support of b is contained in $\Omega_+(\sigma, r_1, r_2)$.

LEMMA 4.4. *Let* $b\in S_+$. *Suppose that* supp b *is contained in* $\Omega_+(\sigma, r_1, r_2)$. *Put*

$$b_1(x,\xi)=b(x,\nabla_x\phi(x,\xi))$$
$$b_2(x,\xi)=b(\nabla_\xi\phi(x,\xi),\xi)$$
$$b_3(x,\xi)=b(x,\eta(x,\xi)).$$

For $R>1$ *large enough, the* b_j*'s belong to* S_+. *More precisely for any* $\varepsilon>0$, *there is* $R=R(\varepsilon,\sigma,r_1,r_2)>0$, *such that the support of* b_j, $j=1,2,3$, *is contained in* $\Omega_+(\sigma-\varepsilon; r_1-\varepsilon, r_2-\varepsilon)$.

Proof. Notice that on the support of b_1, we have

$$\begin{gathered} x\cdot\nabla_x\phi(x,\xi)\geqslant(-1+\sigma)|x|\,|\nabla_x\phi(x,\xi)| \\ |x|\geqslant r_2 \qquad \text{and} \qquad |\nabla_x\phi(x,\xi)|\geqslant r_1. \end{gathered} \tag{4.13}$$

By (4.12) we get from (4.13) that on the support of b_1, $|\xi|\geqslant |\nabla_x\phi(x,\xi)|-CR^{-\varepsilon_0}\geqslant r_1-CR^{-\varepsilon_0}$ and

$$\begin{aligned} x\cdot\xi &\geqslant \mathrm{x}\cdot\nabla_x\phi(x,\xi)-CR^{-\varepsilon_0}|x| \\ &\geqslant(-1+\sigma-4CR^{-\varepsilon_0}(r_1-CR^{-\varepsilon_0})^{-1})|x|\,|\xi|. \end{aligned}$$

This proves the desired result for b_1 by taking R large enough. The other cases can be proved by the same method. ▮

Lemma 4.4 enables us to prove the following result which is important in the application of outgoing and incoming parametrices.

LEMMA 4.5. *Let* $b_+\in S_+$. *Assume that for some* $\sigma_+\in\,]-1,1[$,

$$b_+(x,\xi)=0, \qquad \text{if} \quad \hat{x}\cdot\hat{\xi}<\sigma_+.$$

Then for any $\varepsilon > 0$, there exists $R = R(\varepsilon, b_+)$ such that for every $N \geqslant 1$, one has

$$b_+(x, hD) = I(a_N(h), h)\, I(b(h), h)^* + r_N(x, hD; h),$$

where $a_N(h)$ is the amplitude constructed in Theorem 2.4, $b(h)$ is of the form $b_N(h) = \sum_{j=0}^N h^j b_j$ with $b_j \in S_+$ and $b_j(x, \xi) = 0$ if $\hat{x} \cdot \hat{\xi} < \sigma_+ - \varepsilon$, and $r_N(h)$ is a symbol satisfying

$$|\partial_x^\alpha \partial_\xi^\beta r_N(x, \xi; h)| \leqslant C_{\alpha\beta,N} \langle x \rangle^{-N-|\alpha|} \qquad \text{on} \quad \mathbf{R}^{2n}$$

uniformly in $h \in]0, 1]$.

Proof. Suppose that the support of b_+ is contained in $\Omega_+(\sigma_+, r_1, r_2)$. Let $R = R(\varepsilon; \sigma_+, r_1, r_2)$ be given in Lemma 4.4. Let $\rho \in C^\infty(\mathbf{R}^n)$ such that

$$\rho(x) = 0, \qquad \text{if} \quad |x| \leqslant 3R/2;\ 1, \qquad \text{if} \quad |x| \geqslant 2R.$$

Put $b_{+,1}(x, \xi) = \rho(x)\, b_+(x, \xi)$. Then it is enough to prove the Lemma for $b_{+,1}$ instead of b_+. Let $a_N(h)$ satisfy (i) of Theorem 2.4 in the region $\Omega_+(\varepsilon', E; R)$ with $\varepsilon', E > 0$ sufficiently small. Put $b(h) = \sum_{j=0}^N h^j b_j$ with b_j to be determined. By Lemma A.1, $I(a_N(h), h)\, I(b(h), h)^*$ is a pseudo-differential operator with symbol $C(h)$,

$$C(h) = \sum_{j=0}^N h^j C_j + h^{N+1} r_N(h),$$

with the remainder $r_N(h)$ belonging to S^{-N} and C_j given by

$$C_j(x, \xi) = \sum_{k+l+|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha D_y^\alpha (a_k(x, y, \xi)\, \overline{b_l(x, y, \xi)}\, J(x, y, \xi))|_{y=x},$$
$$j = 0, 1, \ldots, N. \tag{4.14}$$

Here we hae put $a_N(h) = \sum_{k=0}^N h^k a_k$ and

$$a_k(x, y, \xi) = a_k(x, \eta(x, y, \xi)), \qquad k = 0, 1, \ldots, N$$
$$b_j(x, y, \xi) = b_j(x, \eta(x, y, \xi)), \qquad j = 0, 1, \ldots, N.$$

For the definition of $\eta(x, y, \xi)$ and $J(x, y, \xi)$, see Lemma A.1. Therefore we have to show that we can find b_j satisfying the properties of the Lemma such that

$$C_0(x, \xi) = b_{+,1}(x, \xi)$$
$$C_j(x, \xi) = 0, \qquad j = 1, 2, \ldots, N$$

on $\mathbf{R}^{2n}$. Consider first the equation $C_0 = b_{+,1}$. Notice that

$$C_0(x, \xi) = a_0(x, \eta(x, \xi))\, \overline{b_0(x, \eta(x, \xi))}\, J(x, \xi).$$

$\eta(x, \xi)$ is the inverse diffeomorphism of $\xi \to \nabla_x \phi(x, \xi)$ and $J(x, \xi) = |\det(\partial_x \partial_\xi \phi(x, \xi))|^{-1}$. Hence the equation $C_0 = b_{+,1}$ is equivalent to

$$a_0(x, \xi)\, \overline{b_0(x, \xi)} = b_{+,1}(x, \nabla_x \phi(x, \xi))\, |\det(\partial_x \partial_\xi \phi(x, \xi))|, \qquad \text{on} \quad \mathbf{R}^{2n}. \tag{4.15}$$

By the construction of $a_N(h)$, a_0 is of the form

$$a_0(x, \xi) = e^{F(x, \xi)} \Psi(x, \xi)$$

with $\Psi(x, \xi) = 1$ for $(x, \xi) \in \Omega_+(\varepsilon', E; R)$. By the choice of R, the support of $b_+(.,\nabla_x \phi(.,.))$ is contained in $\Omega_+(\sigma - \varepsilon, r_1 - \varepsilon; 3R/2)$. Consequently for $0 < \varepsilon' < \sigma - \varepsilon$ and $0 < E < r_1 - \varepsilon$, $\Psi = 1$ on the support of $b_{+,1}(.,\nabla_x \phi(.,.))$ and the function b_0 defined by

$$\overline{b_0(x, \xi)} = e^{-F(x,\xi)} b_{+,1}(x, \nabla_x \phi(x, \xi))\, |\det(\partial_x \partial_\xi \phi(x, \xi)|$$

satisfies (4.15). Using Proposition 2.3 and Lemma 4.4, one sees that $b_0 \in S_+$ and

$$b_0(x, \xi) = 0, \qquad \text{if} \quad \hat{x} \cdot \hat{\xi} < \sigma - \varepsilon.$$

For $j \geqslant 1$, the equation $C_j = 0$ can be written as

$$a_0(x, \xi)\, \overline{b_j(x, \xi)} = f_j(x, \xi), \tag{4.16}$$

where f_j is determined by $a_0, a_1, \ldots, a_j$ and $b_0, b_1, \ldots, b_{j-1}$ according to (4.14). By an induction, we can show that $f_j \in S_+$ and $\operatorname{supp} f_j \subset \operatorname{supp} b_0$. Therefore,

$$b_j(x, \xi) = \overline{(e^{-F(x,\xi)} f_j(x, \xi))}$$

solves Eq. (4.16). This proves the Lemma by induction. ∎

Now we can give the proof of Theorem 4.2.

Proof of Theorem 4.2. We prove $(4.9)_+$ for $r = 1$. For fixed $s \geqslant 0$, take $N \geqslant 2s + 3$. Applying Lemma 4.5, we have

$$\begin{aligned} &\| \langle x \rangle^{-s-1} \chi(H^h)\, U(t, h)\, b_+(x, hD) \langle x \rangle^s \| \\ &\quad \leqslant \| \langle x \rangle^{-s-1} \chi(H^h)\, U(t, h)\, I(a_N(h))\, I(b(h), h)^* \langle x \rangle^s \| \\ &\qquad + \| \langle x \rangle^{-s-1} \chi(H^h)\, U(t, h)\, r_N(x, hD; h) \langle x \rangle^s \|. \end{aligned}$$

By (4.14), $r_N(x, hD; h)$ is uniformly continuous from $L^{2,-s}$ to $L^{2,1}$. Applying Theorem 4.1 and Lemma 4.3, we get

$$\|\langle x\rangle^{-s-1}\chi(H^h)\, U(t,h)\, b_+(x,hD)\langle x\rangle^s\| \leqslant C(1+t)^{-1} \qquad \text{for} \quad t \geqslant 0$$

uniformly in $h \in]0,1]$. This proves $(4.9)_+$. Notice that the results similar to Lemmas 4.4 and 4.5 are also true for $b \in S_-$. By constructing an incoming parametrix (see Remark (2.5)), we can prove $(4.9)_-$. By the symbolic calculus of h-pseudo-differential operators, if $b_\pm \in S_\pm$, then $b_\pm(x, hD)^*$ is also a h-pseudo-differential operator with the symbol admitting an expansion

$$\sum_{j=0}^{N} h^j d_{\pm,j} + h^{N+1} r_{\pm,N}(h)$$

with $d_{\pm,j} \in S_\pm$ and $\{r_{\pm,N}(h), 0<h\leqslant 1\}$ being bounded in S^{-N}. Thus from $(4.9)_\pm$ it follows that

$$\|\langle x\rangle^{-s-\rho}\chi(H^h)\, U(t,h)\, b_\pm(x,hD)^*\langle x\rangle^s\| \leqslant C(1+|t|)^{-\rho} \qquad \text{for} \quad \pm t \geqslant 0 \tag{4.17$_\pm$}$$

uniformly in $h \in]0,1]$. Taking the adjoint in $(4.17)_\pm$, we get $(4.8)_\pm$. Theorem 4.2 is proved. ∎

Making use of Theorem 4.2, we can establish the time-decay for scattering solutions in two-sided microlocalised form.

THEOREM 4.6. *Let $b_\pm \in S_\pm$. Suppose that there are $\sigma_\pm \in]-1,1[$, $\sigma_- < \sigma_+$, such that*

$$b_+(x,\xi)=0, \qquad \text{if} \quad \hat{x}\cdot\hat{\xi} < \sigma_+$$
$$b_-(x,\xi)=0, \qquad \text{if} \quad \hat{x}\cdot\hat{\xi} > \sigma_-.$$

Then under the assumptions (4.1), *for every $s \geqslant 0$ and $\chi \in C_0^\infty(J)$, one has*

$$\|\langle x\rangle^s b_\mp(x,hD)\, \chi(H^h)\, U(t,h)\, b_\pm(x,hD)\langle x\rangle^s\| \leqslant C(1+|t|)^{-1} \quad \textit{for} \quad \pm t > 0 \tag{4.18$_\pm$}$$

uniformly in $h \in]0,1]$.

Proof. We prove only $(4.18)_+$. Take $b(h) = \sum_{j=0}^N h^j b_j \in S_+$ such that for $R > R(\varepsilon, b_+)$ large enough

$$b_+(x,hD) = I(a_N(h))\, I(b(h),h)^* + r_{N,1}(x,hD;h)$$

with b_j supported in $\{\hat{x}\cdot\hat{\xi}>\sigma_+-\varepsilon\}$ and $\{r_{N,1}(h), 0<h\leqslant 0\}$ a bounded set in S^{-N}. (See Lemma 4.5.) By (4.10) we can write

$$\begin{aligned} &b_-(x, hD)\,\chi(H^h)\,U(t,h)\,b_+(x, hD)\\ &\quad= b_-(x, hD)\,\chi(H^h)\,U(t,h)\,r_{N,1}(x, hD; h)\\ &\qquad+ b_-(x, hD)\,\chi(H^h)\,I(a_N(h), h)\,U_0(t,h)\,I(b(h), h)^*\\ &\qquad+ i\int_0^t b_-(x, hD)\,\chi(H^h)\,U(t-r, h)\,I(P_N(h), h)\,U_0(r,h)\,I(b(h), h^*)\,dr. \end{aligned} \tag{4.19}$$

The first term on the right hand side of (4.19) can be estimated by making use of Theorem 4.2:

$$\begin{aligned} &\|\langle x\rangle^s b_-(x, hD)\,\chi(H^h)\,U(t,h)\,r_{N,1}(x, hD; h)\langle x\rangle^s\|\\ &\quad\leqslant C\|\langle x\rangle^s b_-(x, hD)\,\chi(H^h)\,U(t,h)\langle x\rangle^{-s-1}\|\leqslant C'(1+t)^{-1}, \quad \text{for } t>0 \end{aligned} \tag{4.20}$$

uniformly in $h\in\,]0,1]$, if $0\leqslant s<N/2-1$. For the third term, applying Theorem 4.2, by the method used in the proof of Lemma 4.3, we can show that for $0\leqslant s<N/2-1$,

$$\begin{aligned} &\left\|\langle x\rangle^s\int_0^t b_-(x,hD)\,\chi(H^h)\,U(t-r,h)\,I(P_N(h),h)\,U_0(r,h)\,I(b(h),h)^*\,dr\,\langle x\rangle^s\right\|\\ &\quad\leqslant C\int_0^t \|\langle x\rangle^s b_-(x, hD)\,\chi(H^h)\,U(t-r,h)\langle x\rangle^{-s-1}\|\,(1+r)^{-(N/2-s)}dr\\ &\quad\leqslant C'(1+t)^{-1} \qquad \text{for} \quad t>0 \end{aligned} \tag{4.21}$$

uniformly in $h\in\,]0,1]$. To estimate the second term in (4.19), notice that $b_-(x, hD)\,\chi(H^h)$ is a h-pseudo-differential operator,

$$b_-(x, hD)\,\chi(H^h)=\sum_{k=0}^N h^k\,d_k(x, hD)+h^{N+1}R_{N,2}(h),$$

where $R_{N,2}(h)$ is uniformly continuous from $L^{2,s}$ to $L^{2,s+N}$ for every $s\in\mathbf{R}$ and the support of d_k is contained in that of b_- for $k=0, 1, ..., N$. By Proposition A.3 and a result similar to Lemma 4.4 for $b\in S_-$, we deduce that for every $\varepsilon>0$, there exists $R(\varepsilon, b_-)>0$ such that for $R>R(\varepsilon, b_-)$ we have

$$b_-(x, hD)\,\chi(H^h)\,I(a_N(h), h)=I(e(h), h)+R_{N,3}(h), \tag{4.22}$$

where $e(h)$ is of the form $e(h)=\sum_{j=0}^{N'} h^j e_j$, $e_j \in S_-$ and

$$e_j(x, \xi)=0, \qquad \text{if} \quad \hat{x}\cdot\hat{\xi}>\sigma_- - \varepsilon$$

and $R_{N,3}(h)$ is uniformly continuous from $L^{2,s}$ to $L^{2,s+N}$ for every $s \in \mathbf{R}$. By the assumption $\sigma_- < \sigma_+$, we can take $0<\varepsilon<(\sigma_+ - \sigma_-)/4$. Then $b(h)$ and $e(h)$ are of disjoint support. Applying $(3.17)_+$, we get for every $s \geqslant 0$

$$\|\langle x\rangle^s I(e(h), h)\, U_0(t, h)\, I(b(h), h)^* \langle x\rangle^s\| \leqslant C(1+t)^{-1}, \qquad \text{for} \quad t>0.$$

From (4.22) and Proposition 3.2, we have for $0 \leqslant s < N/2-1$,

$$\|\langle x\rangle^s b_-(x, hD)\, \chi(H^h)\, I(a_N(h), h)\, U_0(t, h)\, I(b(h), h)^*\| \leqslant C(1+t)^{-1} \qquad \text{for} \quad t>0 \tag{4.23}$$

uniformly in $h \in]0, 1]$. Now $(4.18)_+$ result from (4.19), (4.20), (4.21), and (4.23). This proves Theorem 4.6. ▮

5. Proof of Theorem 1

Theorems 4.2 and 4.6 are important in that they enable us to get the best local energy decay for scattering solutions by a simple inductive argument. In this section we retain the conditions (4.1) and we want to prove Theorem 1 for $s \geqslant 0$.

Since $u(t, h)=U(t, h)\, f(h)$, it is clear that Theorem 1 is equivalent to the following.

THEOREM 5.1. *Under assumptions* (4.1), *let* $\chi \in C_0^\infty(J)$. *Then*

(i) *For every* $s \geqslant 0$,

$$\|\langle s\rangle^{-s}\chi(H^h)\, U(t, h)\langle x\rangle^{-s}\| \leqslant C(1+|t|)^{-s} \qquad \textit{for} \quad t \in \mathbf{R}. \tag{5.1}$$

(ii) *Let* $b_\pm \in S_\pm$. *Then every* $s, r \geqslant 0$, *one has*

$$\|\langle x\rangle^r b_\mp(x, hD)\, U(t, h)\, \chi(H^h)\langle x\rangle^{-r-s}\| \leqslant C(1+|t|)^{-s} \qquad \textit{for} \quad \pm t \geqslant 0 \tag*{$(5.2)_\pm$}$$

and

$$\|\langle x\rangle^{-r-s} U(t, h)\, \chi(H^h)\, b_\pm(x, hD)\langle x\rangle^r\| \leqslant C(1+|t|)^{-s} \qquad \textit{for} \quad \pm t \geqslant 0. \tag*{$(5.3)_\pm$}$$

(iii) *Let* $b_\pm \in S_\pm$. *Suppose that there are* $\sigma_\pm \in]-1, 1[$, $\sigma_- < \sigma_+$, *such that*

$$b_+(x, \xi)=0, \qquad \textit{if} \quad \hat{x}\cdot\hat{\xi}<\sigma_+$$

$$b_-(x, \xi)=0, \qquad \textit{if} \quad \hat{x}\cdot\hat{\xi}>\sigma_-.$$

Then for every $s \geqslant 0$, $M \geqslant 0$,

$$\|\langle x\rangle^M b_{\mp}(x, hD)\, U(t, h)\, \chi(H^h)\, b(x, hD)\langle x\rangle^M\| \leqslant C(1+|t|)^{-s}, \qquad \text{for} \quad \pm t \geqslant 0 \qquad (5.4)_{\pm}$$

All these estimates are uniform with respect to $h \in]0, 1]$.

Notice that for $0 \leqslant s \leqslant 1$, Theorem 5.1 is proved in Section 4. See Theorems 4.1, 4.2, and 4.6. We need the following result on functional calculus for h-pseudo-differential operators.

LEMMA 5.2. *Let* $f \in C_0^\infty(]0, +\infty[)$. *Then* $f(H^h)$ *is a pseudo-differential operator. For every* $N \geqslant 0$, $f(H^h)$ *can be written as*

$$f(H^h) = \sum_{j=0}^{N} h^j f_j(x, hD) + h^{N+1} R_N(h)$$

with $R_N(h)$ *uniformly continuous from* $L^{2,s}$ *to* $L^{2,s+N}$ *for every* $s \in \mathbf{R}$ *and* $f_k \in S^{-k}$ *for* $k = 0, 1, ..., N$. *In particular,*

$$f_0(x, \xi) = f(|\xi|^2 + V(x))$$

and the support of f_k, $k = 1, 2, ..., N$, *is contained in that of* $f'(|\xi|^2 + V(x))$.

Lemma 5.2 is a special case of the general results on functional calculus of h-pseudo-differential operators (see [9]). We notice particularly that if $b(z)$ is a symbol defined by

$$b(x, \xi; z) = (|\xi|^2 + V(x) - z)^{-1} \qquad \text{for} \quad z \in C, \operatorname{Im} z \neq 0,$$

by a simple calculus one gets

$$(H^h - z)\, b(x, hD; z) = I + h r_1(x, hD; z) + h^2 r_2(x, hD; z)$$

with $r_1(z)$ and $r_2(z)$ satisfying

$$|\partial_x^\alpha \partial_\xi^\beta r_j(x, \xi; z)| \leqslant C_{\alpha\beta} \langle x\rangle^{-j-|\alpha|-\varepsilon_0} \langle \xi\rangle^{-|\beta|}$$

for fixed $z \in \mathbf{C}$, $\operatorname{Im} z \neq 0$. This enables us to gain the decrease in x in the asymptotic expansion in h. We refer to [9] for the details of the proof.

By Lemma 5.2 we can construct a "partition of unity" for $f(H^h)$.

LEMMA 5.3. *Let* $f \in C_0^\infty(]0, +\infty[)$. *For every* $\sigma \in]-1, 1[$, $\varepsilon > 0$ *and* $N \geqslant 1$, *we can find* $b_{\pm}(h) \in S_{\pm}$ *depending polynomially on h such that*

$$b_+(x, \xi, h) = 0, \qquad \text{if} \quad \hat{x} \cdot \hat{\xi} < \sigma - \varepsilon; \qquad b_-(x, \xi, h) = 0, \qquad \text{if} \quad \hat{x} \cdot \hat{\xi} > \sigma + \varepsilon$$

and that

$$f(H^h)=b_+(x,hD;h)+b_-(x,hD;h)+R_N(h),$$

where $R_N(h)$ is uniformly continuous from $L^{2,s}$ to $L^{2,s+N}$ for every $s\in\mathbf{R}$.

Proof. By Lemma 5.2, it suffices to show that for every k, $0\leqslant k\leqslant N$, we can find $b_\pm$ and $R_N(h)$ so that

$$f_k(x,hD)=b_+(x,hD)+b_-(x,hD)+R_N(h). \tag{5.5}$$

Let $E=\inf\operatorname{supp} f>0$. Since V decreases as $0(|x|^{-\varepsilon_0})$ at infinity, for $R>0$ large enough, $|x|\geqslant R$, we have $|\xi|\geqslant\sqrt{E}/2$ if $\xi\in\operatorname{supp} f_k(x,.)$. Let $\chi\in C_0^\infty(\mathbf{R}^n)$ so that

$$\chi(x)=1, \qquad \text{if} \quad |x|\leqslant 1; \qquad \chi(x)=0, \qquad \text{if} \quad |x|\geqslant 2.$$

Let $\rho\in C^\infty(\mathbf{R})$ so that

$$\rho(t)=1, \qquad \text{if} \quad t\geqslant\sigma+\varepsilon; \qquad \rho(t)=0, \qquad \text{if} \quad t<\sigma-\varepsilon.$$

Put $f_k(x,\xi)=b_+(x,\xi)+b_-(x,\xi)+G(x,\xi)$, where

$$b_+(x,\xi)=(1-\chi(x/R))\,\rho(\hat{x}\cdot\hat{\xi})\,f_k(x,\xi)$$
$$b_-(x,\xi)=(1-\chi(x/R))(1-\rho(\hat{x}\cdot\hat{\xi}))\,f_k(x,\xi)$$
$$G(x,\xi)=\chi(x/R)\,f_k(x,\xi).$$

Then $b_\pm$ belongs to $S_\pm$ and has the desired properties, and $G(x,hD)$ is continuous from $L^{2,s}$ to $L^{2,s+N}$ for every $N\geqslant 0$ and $s\in\mathbf{R}$. This proves (5.5) with $R_N(h)=G(x,hD)$. The Lemma is proved. ∎

Now we are able to give the proof for Theorem 5.1. We will use the micro-local energy decay established in Section 4 and a partition of unity as in Lemma 5.3.

Proof of Theorem 5.1. It is sufficient to prove the following estimates for every $k\in\mathbf{N}$:

$$\|\langle x\rangle^{-k}\chi(H^h)\,U(t,h)\langle x\rangle^{-k}\|\leqslant C(1+|t|)^{-k}, \qquad \text{for} \quad t\in\mathbf{R}. \tag*{$(5.6)_k$}$$

Let $b_\pm\in S_\pm$. Then for any $s\geqslant 0$, one has

$$\|\langle x\rangle^s b_\pm(x,hD)\,U(t,h)\,\chi(H^h)\langle x\rangle^{-s-k}\|\leqslant C(1+|t|)^{-k} \qquad \text{for} \quad \mp t\geqslant 0 \tag*{$(5.7)_{k,\pm}$}$$

and

$$\|\langle x\rangle^{-s-k}\chi(H^h)\,U(t,h)\,b_\pm(x,hD)\langle s\rangle^s\|\leqslant C(1+|t|)^{-k} \qquad \text{for} \quad \pm t>0. \tag*{$(5.8)_{k,\pm}$}$$

Let $b_\pm \in S_\pm$ satisfying the conditions of (iii) of Theorem 5.1. Then for any $s \geqslant 0$, one has

$$\|\langle x\rangle^s b_\mp(x, hD)\,\chi(H^h)\,U(t,h)\,b_\pm(x,hD)\langle x\rangle^s\| \leqslant C(1+|t|)^{-k}, \quad \text{for} \quad \pm t \geqslant 0. \tag{5.9$_{k,\pm}$}$$

We will prove (5.6)–(5.9) by an induction on k. For $k=0$, 1, these results have been proved in Section 4. Suppose now $k \geqslant 2$ and (5.6)–(5.9) are true for $k-1$. We prove first $(5.6)_k$.

Take $\chi_j \in C_0^\infty(J)$, $j=1, 2$, such that $\chi_j = 1$ on the support of χ_{j-1} (with $\chi_0 = \chi$). We write

$$U(t,h)\,\chi(H^h) = \chi_1(H^h)\,U(t/2,h)\,\chi_2(H^h)\,U(t/2,h)\,\chi(H^h). \tag{5.10}$$

By Lemma 5.3, we have the decomposition for $\chi_2(H^h)$:

$$\chi_2(H^h) = P_+(x, hD; h) + P_-(x, hD; h) + R_N(h) \qquad (N \geqslant 2k+1) \tag{5.11}$$

with $P_\pm(h) \in S_\pm$, and $R_N(h)$ is uniformly continuous from $L^{2,s}$ to $L^{2,s+N}$. By $(5.6)_1$ and $(5.6)_{k-1}$, we get

$$\begin{aligned}
&\|\langle x\rangle^{-k}\chi_1(H^h)\,U(t/2,h)\,R_N(h)\,U(t/2,h)\,\chi(H^h)\langle x\rangle^{-k}\| \\
&\quad \leqslant C\|\langle x\rangle^{-1}\chi_1(H^h)\,U(t/2,h)\langle x\rangle^{-1}\|\,\|\langle x\rangle^{-k+1}U(t/2,h)\,\chi(H^h)\langle x\rangle^{-k+1}\| \\
&\quad \leqslant C'(1+|t|)^{-k} \qquad \text{for} \quad t \in \mathbf{R}.
\end{aligned}$$

If $t \geqslant 0$, we write

$$\begin{aligned}
&\chi_1(H^h)\,U(t/2,h)\,P_+(x,hD;h)\,U(t/2,h)\,\chi(H^h) \\
&\qquad = \{\chi_1(H^h)\,U(t/2,h)\,P_+(x,hD;h)\}\,U(t/2,h)\,\chi(H^h).
\end{aligned}$$

If $t \leqslant 0$, we write

$$\begin{aligned}
&\chi_1(H^h)\,U(t/2,h)\,P_+(x,hD;h)\,U(t/2,h)\,\chi(H^h) \\
&\qquad = \chi_1(H^h)\,U(t/2,h)\{P_+(x,hD;h)\,U(t/2,h)\,\chi(H^h)\}.
\end{aligned}$$

In both cases we can apply $(5.6)_1$, $(5.7)_{k-1}$, and $(5.8)_{k-1}$ to get

$$\begin{aligned}
&\|\langle x\rangle^{-k}\chi_1(H^h)\,U(t/2,h)\,P_+(x,hD;h)\,U(t/2,h)\,\chi(H^h)\langle x\rangle^{-k}\| \\
&\quad \leqslant C(1+|t|)^{-k}, \qquad \text{for} \quad t \in \mathbf{R}.
\end{aligned}$$

Similarly we can show

$$\begin{aligned}
&\|\langle x\rangle^{-k}\chi_1(H^h)\,U(t/2,h)\,P_-(x,hD;h)\,U(t/2,h)\,\chi(H^h)\langle x\rangle^{-k}\| \\
&\quad \leqslant C(1+|t|)^{-k}, \qquad \text{for} \quad t \in \mathbf{R}.
\end{aligned}$$

By (5.10) and (5.11), $(5.6)_k$ is proved. To prove $(5.7)_k$, we have to make a special partition of unity in (5.11). Since $b_+ \in S_+$, there exists $\sigma_+ \in]-1, 1[$ such that

$$b_+(x, \xi) = 0, \qquad \text{if} \quad \hat{x} \cdot \hat{\xi} < \sigma_+.$$

For $\varepsilon > 0$ sufficiently small, by Lemma 5.3, we can take $p_-(h) \in S_-$ such that

$$p_-(x, \xi; h) = 0, \qquad \text{if} \quad \hat{x} \cdot \hat{\xi} > \sigma_+ - \varepsilon.$$

This means that we can apply $(5.9)_{k-1}$ to the pair of operators b_+ and $p_-(h)$. As in the proof of $(5.6)_k$, we have the estimate

$$\begin{aligned}
&\|\langle x\rangle^s b_+(x, hD)\, U(t, h)\, \chi(H^h)\langle x\rangle^{-s-k}\| \\
&\quad \leqslant \|\langle x\rangle^s b_+(x, hD)\, \chi_1(H^h)\, U(t/2, h)\langle x\rangle^{-s-1}\| \\
&\qquad \times \|\langle x\rangle^{s+1} P_+(x, hD; h)\, U(t/2, h)\, \chi(H^h)\langle x\rangle^{-s-k}\| \\
&\qquad + \|\langle x\rangle^s b_+(x, hD)\, \chi_1(H^h)\, U(t/2, h)\, P_-(x, hD; h)\langle x\rangle^{s+k}\| \\
&\qquad \times \|\langle x\rangle^{-s-k} \chi(H^h)\, U(t/2, h)\langle x\rangle^{-s-k}\| \\
&\qquad + \|\langle x\rangle^s b_+(x, hD)\, \chi_1(H^h)\, U(t/2, h)\, R_N(h)\, U(t/2, h)\, \chi(H^h)\langle x\rangle^{-s-k}\| \\
&\quad \leqslant C(1 + |t|)^{-k}, \qquad \text{for} \quad t \leqslant 0.
\end{aligned} \tag{5.12}$$

Here we have applied $(5.7)_{1,+}$ and $(5.7)_{k-1,+}$ to the first term in the middle of (5.12), $(5.9)_{1,-}$ and $(5.6)_{k-1}$ to the second term; $(5.7)_{1,+}$ and $(5.6)_{k-1}$ to the third term. This proves $(5.7)_{k,+}$. In the same way, we can prove $(5.7)_{k,-}$. Relation $(5.8)_k$ follows from $(5.7)_k$ by taking the adjoint; $(5.9)_k$ can be proved by the same method. The only difference is that this time we have to choose $P_\pm(h)$ in (5.11) such that $P_+(h)$ and b_- are of disjoint support and the same is true for $P_-(h)$ and b_+. This is possible, because by the assumption on $b_\pm$, we have $\sigma_- < \sigma_+$. Then it is sufficient to apply Lemma 5.3 with $\sigma = (\sigma_+ + \sigma_-)/2$ and $\varepsilon = (\sigma_+ - \sigma_-)/4 > 0$. By induction, we have finished the proof of (5.6)–(5.9). Now Theorem 5.1 follows by a simple interpolation. This proves Theorem 5.1, hence also Theorem 1. ∎

For fixed $h > 0$, applying the estimate (4.7) and proceeding as before, we can prove the following.

THEOREM 5.4. *Let V be a short range potential satisfying* (1.1) *for some* $\varepsilon_0 > 1$. *Let $f \in C^\infty(\mathbf{R}_+)$ with bounded derivatives on $\mathbf{R}_+$ and* $\inf \operatorname{supp} f > 0$. *Then for every $s \geqslant 0$, one has*

$$\|\langle x\rangle^{-s} f(H)\, U(t)\langle x\rangle^{-s}\| \leqslant C(1 + |t|)^{-s}, \qquad t \in \mathbf{R}. \tag{5.13}$$

Let $b_\pm \in S_\pm$. For every $\rho \geqslant 0$,

$$\|\langle x\rangle^\rho b_\pm(x, D)\, U(t)\, f(H)\langle x\rangle^{-\rho-s}\| \leqslant C(1+|t|)^{-s}, \qquad \text{for} \quad \mp t \geqslant 0 \tag{5.14}$$

and

$$\|\langle x\rangle^{-\rho-s} f(H^h)\, U(t)\, b_\pm(x, D)\langle x\rangle^\rho\| \leqslant C(1+|t|)^{-s}, \qquad \text{for} \quad \pm t \geqslant 0. \tag{5.15}$$

Let $b_\pm \in S_\pm$ satisfy the conditions of (iii) *of Theorem* 6.2. *We have, for every $M \geqslant 0$,*

$$\|\langle x\rangle^M b_\mp(x, D)\, f(H)\, U(t)\, b_\pm(x, D)\langle x\rangle^M\| \leqslant C(1+|t|)^{-s} \quad \text{for} \ \pm t \geqslant 0. \tag{5.16}$$

6. Semiclassical Micro-local Resolvent Estimates

From now on, the assumption on the potential V is the following:

$$V \text{ is a long range potential satisfying (1.1) and (1.4).} \tag{6.1}$$

In this section we will give some estimates over the resolvent $R(z, h)$ microlocalised in outgoing or incoming regions. For fixed h, this type of result is known [11] and can be applied to study the singularity of scattering amplitude.

In order to use the outgoing (resp. incoming) h-parametrix constructed in Section 2, we need a result on the free resolvent $R_0(z, h)$ microlised by Fourier integral operators. Recall that the phase function ϕ depends on a parameter $R > 0$ (see (2.4)).

Proposition 6.1. *For every $b_\pm \in S_\pm$, there exists $R = R(b_+) > 0$ such that for every $s > 1/2$, one has*

$$\|\langle x\rangle^{s-1} I(b_\pm, h)\, R_0(\lambda \mp i0; h)\langle x\rangle^{-s}\| \leqslant Ch^{-1} \tag{6.2$_\pm$}$$

and

$$\|\langle x\rangle^{-s} R_0(\lambda \pm i0; h)\, I(b_\pm, h)^*\langle x\rangle^{s-1}\| \leqslant Ch^{-1} \tag{6.3}$$

uniformly in $h \in\,]0, 1]$ and locally uniformly in $\lambda > 0$.

Proof. Let $b_+ \in S_+$. By Lemma 4.4, there exists $a \in S_+$ such that for $R > 1$ large enough, $a(\nabla_\xi \phi(x, \xi), \xi) = 1$ on the support of b_+. Applying Proposition A.4 and taking notice that the derivatives of $a(\nabla_\xi \phi(x, \xi), \xi)$ are equal to zero on the support of b_+, we get for every $N \geqslant 1$

$$I(b_+, h)\, a(x, hD) = I(b_+, h) + h^N I(r_N(h), h) \tag{6.4}$$

with $\{r_N(h), 0<h\leqslant 1\}$ bounded in S^{-N}. According to Proposition A.2, for $s>1/2$ and $2s\leqslant N$, we get

$$\|\langle x\rangle^{s-1} I(r_N(h), h)\, R_0(z, h)\langle x\rangle^{-s}\| \leqslant Ch^{-1}$$

for $h\in]0, 1]$ and $z\in C$ with $\operatorname{Im} z\neq 0$ and $\operatorname{Re} z>0$. Since $I(b_+, h)$ is uniformly continuous from $L^{2,s-1}$ to $L^{2,s-1}$, it follows from (6.4) and Theorem 3.4 that

$$\begin{aligned}&\|\langle x\rangle^{s-1} I(b_+, h)\, R_0(z, h)\langle x\rangle^{-s}\|\\ &\quad\leqslant C\|\langle x\rangle^{s-1} a(x, hD)\, R_0(z, h)\langle x\rangle^{-s}\| + C'h^{N-1}\\ &\quad\leqslant C''h^{-1}\end{aligned}$$

for $h\in]0, 1]$ and $\operatorname{Re} z>0$, $\operatorname{Im} z\neq 0$. This proves $(6.2)_+$. The other cases can be proved similarly. ∎

For I an interal in $\mathbf{R}$, we denote $I_\pm=\{z\in C;\ \operatorname{Re} z\in I,\ \pm\operatorname{Im} z\in]0, 1]\}$. $I_0=I_+\cup I_-$. By Proposition 6.1, we can prove an analogue of Theorem 3.4 for the resolvent $R(z, h)$.

THEOREM 6.2. *Let $b_\pm\in S_\pm$. Under the assumptions* (6.1), *let $I\Subset J$. Then for every $s>1/2$, one has*

$$\|\langle x\rangle^{s-1} b_\mp(x, hD)\, R(z; h)\langle x\rangle^{-s}\| \leqslant Ch^{-1}, \qquad \text{for}\quad h\in]0, 1], z\in I_\pm \tag{6.5$_\pm$}$$

and

$$\|\langle x\rangle^{-s} R(z, h)\, b_\pm(x, hD)\langle x\rangle^{s-1}\| \leqslant Ch^{-1}, \qquad \text{for}\quad h\in]0, 1], z\in I_\pm. \tag{6.6}$$

Proof. As in the proof of Theorem 4.2, we will use the outgoing (incoming) parametrix constructed in Section 2 to show the theorem.

Let $b_+\in S_+$ and $s>1/2$ be fixed. Take ε, $E>0$ sufficiently small so that the support of b_+ is contained in $\Omega_+(\varepsilon, E)$. Construct an outgoing parametrix in the region $\Omega_+(\varepsilon, E, R)$ as in Section 2. Then we have the relation, for $N>2s+1$,

$$U(t, h)\, I(a_N(h), h)\, I(b, h)^* = U_N(t, h) + i\int_0^t U(t-r, h)\, R_N(r, h)\, dr, \tag{6.7}$$

where

$$\begin{aligned}U_N(t, h) &= I(a_N(h), h)\, U_0(t, h)\, I(b, h)^*\\ R_N(t, h) &= I(p_N(h), h)\, U_0(t, h)\, I(b, h)^*.\end{aligned}$$

(See Theorem 2.4 for the notations.) By Lemma 4.5, for $R > o$ large enough, there exists $b = b(h)$ belonging to S_+ such that

$$I(a_N(h), h)\, I(b(h), h)^* = b_+(x, hD) + Q_N(h), \tag{6.8}$$

where $Q_N(h)$ is uniformly continuous from $L^{2,\tau}$ to $L^{2,\tau+N}$, $\forall \tau \in \mathbf{R}$. Multiplying (6.7) by $e^{ih^{-1}tz}$, $\operatorname{Im} z > 0$, and integrating over $\mathbf{R}^+$, we get a resolvent equation:

$$\begin{aligned} &R(z, h)\, b_+(x, hD) \\ &\quad = R(z, h)\, Q_N(h) + I(a_N(h), h)\, R_0(z, h)\, I(b(h), h)^* \\ &\qquad + hR(z, h)\, I(p_N(h), h)\, R_0(z, h)\, I(b(h), h)^* \qquad \text{for} \quad \operatorname{Im} z > 0. \end{aligned} \tag{6.9}$$

Recall that it is proved in [20] that under the conditions (6.1) we have

$$\| \langle x \rangle^{-s} R(z, h) \langle x \rangle^{-s} \| \leqslant Ch^{-1}, \qquad \text{for} \quad h \in \,]0, 1]; z \in I_0. \tag{6.10}$$

Hence we get

$$\| \langle r \rangle^{-s} R(z, h)\, Q_N(h) \langle x \rangle^{s-1} \| \leqslant Ch^{-1}, \qquad \text{for} \quad h \in \,]0, 1], z \in I_0.$$

Applying Proposition 6.1, we deduce that

$$\| \langle x \rangle^{-s} I(a_N(h), h)\, R_0(z, h)\, I(b(h), h)^* \langle x \rangle^{s-1} \| \leqslant Ch^{-1}, \;\; h \in \,]0, 1], z \in I_+.$$

To estimate the last term in (6.9), we proceed as in Lemma 4.3. Assume that the support of $b(h)$ is contained in $\Omega_+(2\varepsilon, 2E)$. Choose $\rho_1 \in C^\infty(\mathbf{R})$ such that

$$\rho_1(t) = \begin{cases} 1 & \text{if} \quad t \leqslant -1 + \varepsilon \\ 0 & \text{if} \quad t \geqslant -1 + 3\varepsilon/2. \end{cases}$$

Put $\rho_2(t) = 1 - \rho_1(t)$ and $p_{N,j}(x, \xi, h) = \rho_j(\hat{x} \cdot \hat{\xi})\, p_N(x, \xi; h)$. Then b_+ and $p_{N,1}(h)$ are of disjoint support and $p_{N,2}(h)$ satisfies

$$|\partial_x^\alpha \partial_\xi^\beta p_{N,2}(x, \xi; h)| \leqslant C_{\alpha\beta} \langle x \rangle^{-N-|\alpha|} \qquad (x, \xi) \in \mathbf{R}^{2n}.$$

Consequently, applying Corollary 3.6, we have

$$\| \langle x \rangle^s I(p_{N,1}(h), h)\, R_0(z, h)\, I(b(h), h)^* \langle x \rangle^{s-1} \| \leqslant Ch^{-1}, \qquad z \in I_+.$$

Since $I(p_{N,1}(h), h)$ is uniformly continuous from $L^{2,-s}$ to $L^{2,s}$, we get from Proposition 6.1 that

$$\| \langle x \rangle^s I(p_{N,2}(h), h)\, R_0(z, h)\, I(b(h), h)^* \langle x \rangle^{s-1} \| \leqslant Ch^{-1}$$

for $z \in I_+$. By (6.10), we have proved that

$$\|\langle x\rangle^{-s} R(z, h)\, I(p_N(h), h)\, R_0(z, h)\, I(b(h), h)^* \langle x\rangle^{s-1}\| \leqslant Ch^{-1}, \quad h \in]0, 1]$$

for $z \in I_+$. Now $(6.5)_+$ results easily from (6.9). Relation $(6.5)_-$ can be proved by constructing an incoming parametrix and (6.6) follows by taking the adjoint in (6.5). Theorem 6.2 is proved. ∎

For two-sided localised resolvent estimates, we have the following.

THEOREM 6.3. *Let $b_\pm \in S_\pm$ such that there are $\sigma_\pm \in]-1, 1[$ with $\sigma_- < \sigma_+$ and*

$$b_+(x, \xi) = 0 \qquad \text{if} \quad \hat{x} \cdot \hat{\xi} < \sigma_+$$
$$b_-(x, \xi) = 0 \qquad \text{if} \quad \hat{x} \cdot \hat{\xi} > \sigma_-.$$

Under the assumptions (6.1), *let $I \Subset J$ be a compact interval of non-trapping energy. Then for every $s \geqslant 0$, we have*

$$\|\langle x\rangle^s b_\mp(x, hD)\, R(z; h)\, b_\pm(x, hD) \langle x\rangle^s\| \leqslant Ch^{-1} \qquad (6.11)_\pm$$

for $z \in I_\pm$ and $h \in]0, 1]$.

Proof. From (6.7), it follows that

$$\begin{aligned} R(z, h)\, I(a_N(h), h)\, I(b, h)^* &= I(a_N(h), h)\, R_0(z, h)\, I(b, h)^* \\ &\quad + hR(z, h)\, I(p_N(h), h)\, R_0(z, h)\, I(b, h)^*. \end{aligned} \qquad (6.12)$$

By Proposition A.3, $b_-(x, hD)\, I(a_N(h), h)$ is a Fourier integral operator with the amplitude $d(h)$,

$$d(h) = \sum_{j=0}^{N} h^j d_j + h^{N+1} r_{N,1}(h),$$

where the d_j's can be calculated by the formulae in Proposition A.3 and $\{r_{N,1}(h), 0 < h \leqslant 1\}$ is bounded in S^{-N}. According to Lemma 4.5 we can choose $b = b(h) \in S_+$ such that

$$I(a_N(h), h)\, I(b(h), h)^* = b_+(x, hD) + R_{N,2}(h), \qquad (6.13)$$

where $b(h)$ and d_j are of disjoint support. This is possible for R large enough because of the hypothesis on $b_\pm$. Applying Corollary 3.6, we get

$$\|\langle x\rangle^s I(d_j, h)\, R_0(z, h)\, I(b(h), h)^* \langle x\rangle^s\| \leqslant Ch^{-1}$$

uniformly in $z \in I_+$. From Proposition 6.1, it follows easily

$$\|\langle x\rangle^s b_-(x, hD)\, I(a_N(h), h)\, R_0(z, h)\, I(b(h), h)^* \langle x\rangle^s\| \leqslant Ch^{-1}, \qquad h \in]0, 1], z \in I_+$$

for $0 \leqslant s \leqslant N/2$. Applying Theorem 6.2, we can prove as in Theorem 6.2 that for $0 \leqslant s \leqslant N/2$,

$$\|\langle x\rangle^s b_-(x, hD)\, R(z, h)\, I(p_N(h), h)\, R_0(z, h)\, I(b(h), h)^* \langle x\rangle^s\| \leqslant Ch^{-2}$$

uniformly in $z \in I_+$. From (6.12) and (6.13), we can derive that

$$\|\langle x\rangle^s b_-(x, hD)\, R(z, h)\, b_+(x, hD)\langle x\rangle^s\| \leqslant Ch^{-1}, \qquad h \in]0, 1]$$

for $0 \leqslant s \leqslant N/2$, uniformly in $z \in I_+$. Since $N \geqslant 1$ is arbitrary, this proves $(6.11)_+$. Theorem 6.3 is proved. ∎

7. Proof of Theorem 2

With the micro-local resolvent estimates obtained in the preceding section, we can establish the semiclasssical bounds for the powers of resolvent. Notice that for every fixed $h > 0$, the smoothness of the boundary values for the resolvent is known [13, 14]. Our aim is to give the precise semiclassical estimates, which should be useful in semiclassical scattering theory (see [23, 24]).

Proof of Theorem 2. We use the same method as the proof for Theorem 1. Notice that for $m = 1$, Theorem 2 is proved (see (6.10), Theorems 6.2 and 6.3). Suppose the results are true for $m-1$, $m \geqslant 2$. We want to prove it for m. Let χ be a C^∞ function on $\mathbf{R}$, with bounded derivatives and $\chi(t) = 0$ in a small neighborhood of I in $\mathbf{R}$. Put

$$g(t, z) = \chi(t)(t-z)^{-m}, \qquad t \in \mathbf{R}, \operatorname{Re} z \in I.$$

Then $g(., z) \in C^\infty(\mathbf{R})$ and

$$\left|\frac{\partial^k}{\partial t^k} g(t, z)\right| \leqslant C_k \qquad \text{for} \quad t \in \mathbf{R}, \operatorname{Re} z \in I.$$

Therefore by Lemma 5.2, $g(H^h, z) = \chi(H^h)\, R(z, h)^m$ is a pseudo-differential operator and for every $s \in \mathbf{R}$

$$\sup_{\substack{h \in]0,1] \\ z \in C, \operatorname{Re} z \in I}} \|g(H^h, z)\|_{\mathscr{L}(L^{2,s}, L^{2,s})} < +\infty.$$

This shows that for every $s \geqslant 0$, one has

$$\|\langle x\rangle^{-s}\chi(H^h)(R(z,h))^m\langle x\rangle^{-s}\| \leqslant C$$
$$\|\langle x\rangle^{s-1} b_\pm(x,hD)\,\chi(H^h)(R(z,h))^m\langle x\rangle^{-s}\| \leqslant C$$

and

$$\|\langle x\rangle^{-s}(R(z,h))^m\chi(H^h)\,b_\pm(x,hD)\langle x\rangle^{s-1}\| \leqslant C, \qquad \text{for} \quad h\in\,]0,1]$$

uniformly in $z\in C$ with $\operatorname{Re} z\in I$. If $b_\pm$ satisfies the hypothesis of (iii), applying Lemma 5.2, we can show by symbolic calculus for pseudo-differential operators that $b_+(x,hD)\,g(H^h,z)\,b_-(x,hD)$ is uniformly continuous from $L^{2,-s}$ to $L^{2,s}$ and for every $N\in\mathbf{N}$,

$$\|\langle x\rangle^s b_\pm(x,hD)\,g(H^h,z)\,b_\mp(x,hD)\langle x\rangle^s\| \leqslant C_{s,N}h^N, \qquad h\in\,]0,1]$$

uniformly in $z\in\mathbf{C}$ with $\operatorname{Re} z\in I$. Consequently in order to show the theorem, it is enough to prove that for $f\in C_0^\infty(J)$, $f=1$ in a small neighborhood of I, we have, for $s>m-1/2$,

$$\|\langle x\rangle^{-s}f(H^h)(R(z,h))^m\langle x\rangle^{-s}\| \leqslant C_{m,s}h^{-m} \tag{7.1}$$
$$\|\langle x\rangle^{s-m} b_\pm(x,hD)\,f(H^h)(R(z,h))^m\langle x\rangle^{-s}\| \leqslant C_{m,s}h^{-m}, \qquad z\in I_\mp \qquad (7.2)_\pm$$

and

$$\|\langle x\rangle^{-s}f(H^h)(R(z,h))^m b_\pm(x,hD)\langle x\rangle^{s-m}\| \leqslant C_{m,s}h^{-m}, \qquad z\in I_\pm \qquad (7.3)_\pm$$

and if $b_\pm$ satisfies the conditions of (iii), then

$$\|\langle x\rangle^s b_\mp(x,hD)(R(z,h))^m f(H^h)\,b_\pm(x,hD)\langle x\rangle^s\| \leqslant C_{m,s}h^{-m}, \qquad z\in I_\pm. \qquad (7.4)_\pm$$

According to Lemma 5.3, we have a decomposition for $f(H^h)$ with $N\geqslant 2s+1$,

$$f(H^h)=p_+(x,hD;h)+p_-(x,hD;h)+R_N(h), \tag{7.5}$$

where $p_\pm(h)\in S_\pm$ and $R_N(h)$ is uniformly continuous from $L^{2,r}$ to $L^{2,r+N}$, $\forall r\in\mathbf{R}$.

Proof of (7.1). By (7.5), $(R(z,h))^m f(H^h)$ may be written as

$$\begin{aligned}(R(z,h))^m f(H^h) &= (R(z,h))^{m-1} f(H^h)\,R(z,h)\\ &= (R(z,h))^{m-1}(p_+(x,hD;h)+p_-(x,hD;h)\\ &\quad + R_N(h))\,R(z,h).\end{aligned} \tag{7.6}$$

Applying Theorem 2 for $m-1$, we deduce from (i) that for $s>m-1-1/2$,

$$\|\langle x\rangle^{-s}(R(z,h))^{m-1}R_N(h)\,R(z,h)\langle x\rangle^{-s}\|\leqslant Ch^{-m},\qquad h\in\,]0,1]$$

uniformly in $z\in I_0$. From (i) and (ii) for $m-1$, we conclude as in the proof of Theorem 5.1 that for $s>m-1/2$,

$$\|\langle x\rangle^{-s}(R(z,h))^{m-1}p_+(x,hD;h)\,R(z,h)\langle x\rangle^{-s}\|\leqslant Ch^{-m}$$

and

$$\|\langle x\rangle^{-s}(R(z,h))^{m-1}p_-(x,hD;h)\,R(z,h)\langle x\rangle^{-s}\|\leqslant Ch^{-m}$$

uniformly in $h\in\,]0,1]$ and $z\in I_0$. This proves (7.1). ∎

Proof of $(7.2)_+$. By Lemma 5.3, we can take a decomposition (7.5) for $f(H^h)$ such that $p_-(h)$ and b_+ are of disjoint support and satisfy the conditions of (iii). Making use of the expression (7.6), we can deduce $(7.2)_+$ from the inductie hypothesis as in the proof of (7.1).

Expression $(7.2)_-$ can be proved by the same method and (7.3) follows from (7.2) by taking the adjoint. To prove (7.4), it suffices to notice that under the conditions of (iii), we can choose $p_\pm(h)$ in (7.6) such that the pairs of operators $p_+(h)$ and b_-, $p_-(h)$ and b_+ satisfy still the assumptions of (iii) of Theorem 2 (see Lemma 5.3). This finishes the proof of Theorem 2 by induction. ∎

By the same argument as the proof of Theorem 2, we can show that the boundary values $(R(\lambda\pm i0,h))^m$ exist in corresponding weighted-L^2 spaces and it can be easily verified that they are equal to the $(m-1)$th derivative of $(1/(m-1)!)\;R(\lambda\pm i0,h)$ with respect to λ. As usual, the smoothness of resolvent gives results on time-decay for wave functions. The semiclassical estimates on the derivatives of boundary values of the resolvent lead to the time-decay uniform with respect to $h\in\,]0,1]$.

COROLLARY 7.1. *Under the assumptions of Theorem 2, let $\chi\in C_0^\infty(J)$. Then one has for every $s,\rho>0$ and for any $0<\varepsilon<s$,*

$$\|\langle x\rangle^{-s}\chi(H^h)\,U(t,h)\langle x\rangle^{-s}\|\leqslant C_{s,\varepsilon}h^{-\varepsilon}(1+|t|)^{-s+\varepsilon},\qquad t\in\mathbf{R}\tag{7.7}$$

$$\|\langle x\rangle^{\rho}b_\mp(x,hD)\,\chi(H^h)\,U(t,h)\langle x\rangle^{-\rho-s}\|\leqslant C_{\rho,s;\varepsilon}h^{-\varepsilon}(1+|t|)^{-s+\varepsilon},\quad \textit{for}\quad \pm t>0\qquad (7.8)_\pm$$

$$\|\langle x\rangle^{-\rho-s}\chi(H^h)\,U(t,h)\,b_\pm(x,hD)\langle x\rangle^{\rho}\|\leqslant C_{\rho,s;\varepsilon}h^{-\varepsilon}(1+|t|)^{-s-\varepsilon},\quad \textit{for}\quad \pm t>0.\qquad (7.9)_+$$

If $b_\pm$ satisfies the conditions of (iii) *of Theorem* 2, *then*

$$\|\langle x\rangle^\rho b_\mp(x, hD)\,\chi(H^h)\,U(t,h)\,b_\pm(x, hD)\langle x\rangle^\rho\| \leqslant C_{\rho,s;\varepsilon} h^{-\varepsilon}(1+|t|)^{-s-\varepsilon}, \quad \text{for} \quad \pm t>0. \tag{7.10}$$

The proof of Corollary 7.1 is routine (see [14]). We omit it here.

8. Proof of Theorem 3

In this section we will show how to improve Corollary 7.1 to get Theorem 3. By the arguments used in the proof of Theorem 1 (see Section 5), it suffices to prove the theorem for $s=1$.

Proposition 8.1. *Under the assumptions of Theorem* 3, *we have, for any* $\varepsilon>0$,

$$\|\langle x\rangle^{-1}\chi(H^h)\,U(t,h)\langle x\rangle^{-1}\| \leqslant C_\varepsilon h^{-\varepsilon}(1+|t|)^{-1}, \qquad h\in\,]0,1],\ t\in\mathbf{R}. \tag{8.1}$$

Here $\chi\in C_0^\infty(J)$.

Proof. We want to use the commutator relation (4.3). Take $\chi_1, \chi_2\in C_0^\infty(J)$ such that $2H^h\chi_1(H^h)\,\chi_2(H^h)=\chi(H^h)$. Then from (4.3) we get, for $t\neq 0$,

$$\begin{aligned}\langle x\rangle^{-1}&\chi(H^h)\,U(t,h)\langle x\rangle^{-1}\\ &=\frac{1}{t}\Big\{\langle x\rangle^{-1}\chi_1(H^h)[A^h, U(t,h)]\,\chi_2(H^h)\langle x\rangle^{-1}\\ &\quad-\langle x\rangle^{-1}\int_0^t \chi_1(H^h)\,U(t-s;h)\,\tilde{V}U(s,h)\,\chi_2(H^h)\langle x\rangle^{-1}\,ds\Big\}.\end{aligned} \tag{8.2}$$

We can easily check that $\langle x\rangle^{-1}\chi_1(H^h)\,A^h$ and $A^h\chi_2(H^h)\langle x\rangle^{-1}$ extend to bounded operators on $L^2(\mathbf{R}^n)$. Therefore we have

$$\|\langle x\rangle^{-1}\chi_1(H^h)[A^h, U(t,h)]\,\chi_2(H^h)\langle x\rangle^{-1}\| \leqslant C<+\infty \tag{8.3}$$

uniformly in $h\in\,]0,1]$, $t\in\mathbf{R}$. To treat the remainder we choose $\chi_3\in C_0^\infty(J)$ such that $\chi_3\chi_2=\chi_2$. Applying Lemma 5.3 to χ_3, we get

$$\chi_3(H^h)=b_+(x, hD; h)+b_-(x, hD; h)+R(h)$$

with $R(h)$ uniformly bounded from $L^{2,-1}$ to $L^{2,1}$. Inserting this expression between $\tilde{V}$ and $U(s,h)$, we get three corresponding terms which can be estimated as follows. To fix ideas, let $t>0$. Then from (7.7), $(7.8)_+$, and the fact that $\tilde{V}$ decreases as $0(|x|^{-\varepsilon_0})$, we derive that

$$\left\| \int_0^t \langle x\rangle^{-1} \chi_1(H^h)\, U(t-s;h)\, \tilde{V} b_-(x, hD; h)\, U(s;h)\, \chi_2(H^h) \langle x\rangle^{-1}\, ds \right\|$$

$$\leqslant C \int_0^t \|\langle x\rangle^{-\varepsilon_0} \chi_1(H^h)\, U(t-s;h) \langle x\rangle^{-\varepsilon_0}\|$$

$$\times \|b_-(x, hD; h)\, U(s,h)\, \chi_2(H^h)\langle x\rangle^{-1}\|\, ds$$

$$\leqslant C_\varepsilon h^{-2\varepsilon} t^{-\varepsilon_0+2\varepsilon}, \qquad \text{for any} \quad \varepsilon>0. \tag{8.4}$$

Since the commutator $i[\tilde{V}, b_+(x, hD; h)]$ is uniformly continuous from $L^{2,r}$ to $L^{2,r+1+\varepsilon_0}$, we have a similar result for the term corresponding to $b_+(x, hD; h)$:

$$\left\| \int_0^t \langle x\rangle^{-1} \chi_1(H^h)\, U(t-s;h)\, \tilde{V} b_+(x, hD; h)\, U(s,h)\, \chi_2(H^h) \langle x\rangle^{-1}\, ds \right\|$$

$$\leqslant C_\varepsilon h^{-2\varepsilon} t^{-\varepsilon_0+2\varepsilon}. \tag{8.5}$$

Here we have used (7.7) and $(7.9)_+$. The term corresponding to $R(h)$ can be estimates by (7.7). Hence from (8.2)–(8.5), we get

$$\|\langle x\rangle^{-1} \chi(H^h)\, U(t,h) \langle x\rangle^{-1}\| \leqslant C_\varepsilon((1+|t|)^{-1} + h^{-2\varepsilon}(1+|t|)^{-1-\varepsilon_0/2}). \tag{8.6}$$

This proves (8.1). ∎

The estimate (8.6) shows that for every $\varepsilon>0$, we have

$$\|\langle x\rangle^{-1} \chi(H^h)\, U(t,h) \langle x\rangle^{-1}\| \leqslant C(1+|t|)^{-1},$$

for $h \in]0,1]$ and $|t| \geqslant C' h^{-\varepsilon}$. Repeating the arguments of Section 4 and making use of (8.1) instead of (4.2), we can prove the following.

PROPOSITION 8.2. *Under the assumptions of Theorem 3, let $\chi \in C_0^\infty(J)$. Then for every $s \geqslant 0$ and $r \in [0,1]$, we have, for any $\varepsilon>0$,*

$$\|\langle x\rangle^{s} b_\pm(x, hD)\, U(t,h)\, \chi(H^h) \langle x\rangle^{-s-r}\| \leqslant C_{s,r,\varepsilon} h^{-\varepsilon}(1+|t|)^{-r}, \qquad \mp t>0. \tag{8.7}$$

$$\|\langle x\rangle^{-s-r} \chi(H^h)\, U(t,h)\, b_\pm(x, hD) \langle x\rangle^{-s}\| \leqslant C_\varepsilon h^{-\varepsilon}(1+|t|)^{-r}, \qquad \pm t>0. \tag{8.8}$$

In addition if $b_\pm \in S_\pm$ is chosen so that $\sigma_- < \sigma_+$, we have for every $s \geqslant 0$,

$$\|\langle x\rangle^{-s} b_\mp(x, hD)\, \chi(H^h)\, U(t,h)\, b_\pm(x, hD) \langle x\rangle^{s}\| \leqslant C_\varepsilon h^{-\varepsilon}(1+|t|)^{-1}, \qquad \pm t>0. \tag{8.9}$$

Now applying Propositions 8.1 and 8.2, we can prove Theorem 3 by the inductive argument used in Section 5. We omit the details here.

APPENDIX: A CLASS OF FOURIER INTEGRAL OPERATORS

We collect in this Appendix some basic results on the continuity and the composition of a class of Fourier integral operators. Since the methods are now well known [2, 7, 8] we content ourselves to give a sketch for the proofs.

Let ϕ be a real smooth function on $\mathbf{R}^{2n}$ such that for some $\varepsilon_0 > 0$,

$$|\partial_x^\alpha \partial_\xi^\beta(\phi(x, \xi) - x \cdot \xi)| \leqslant C_{\alpha\beta} \langle x \rangle^{1-\varepsilon_0-|\alpha|} \langle \xi \rangle^{-1} \tag{A.1}$$

$$|(\partial_{x_j} \partial_{\xi_j} \phi(x, \xi)) - I| < 1/2, \qquad \text{on} \quad \mathbf{R}^{2n}. \tag{A.2}$$

Let S^m be the class of symbols introduced in Section 3. For $a \in S^m$, let $I(a, h)$ be the Fourier integral operator associated with amplitude a and phase ϕ by the formula (2.16). Remark that we can write $\phi(x, \xi) - \phi(y, \xi) = (x - y) \cdot \nabla\phi(x, y, \xi)$, where

$$\nabla\phi(x, y, \xi) = \int_0^1 \nabla_x \phi(y + \theta(x - y), \xi)\, d\theta.$$

By (A.2) for every $(x, y) \in \mathbf{R}^{2n}$, the mapping $\xi \to \nabla\phi(x, y, \xi)$ is a global diffeomorphism on $\mathbf{R}^\eta_\xi$. We denote by $\eta = \eta(x, y, \xi)$ the inverse diffeomorphism.

LEMMA A.1. *Let $a_j \in S^{m_j}$ for $j = 1, 2$. Then $B(h) = I(a_1, h)\, I(a_2, h)^*$ is a pseudo-differential operator associated with symbol $b(h)$ having an asymptotic expansion in $h \in\,]0, 1]$. For every $N \geqslant 0$, one has*

$$b(x, \xi; h) = \sum_{j=0}^{N} h^j b_j(x, \xi) + h^{N+1} r_N(x, \xi, h), \tag{A.3}$$

where

$$b_j(x, \xi) = \sum_{|\alpha| = j} \frac{1}{\alpha!} \partial_\xi^\alpha D_y^\alpha(a_1(x, y, \xi)\, \overline{a_2(x, y, \xi)}\, J(x, y, \xi))|_{y=x}$$

$$a_1(x, y, \xi) = a_1(x, \eta(x, y, \xi))$$

$$a_2(x, y, \,) = a_2(y, \eta(x, y, \xi))$$

and $J(x, y, \xi)$ is the Jacobian of $\eta(x, y, \xi)$ with respect to ξ, $\{r_N(h), h \in [0, 1]\}$ is a bounded set in $S^{m_1 + m_2 - N - 1}$.

Proof. By a simple calculation, we see that $I(a_1, h)\, I(a_2, h)^*$ is a pseudo-differential operator with symbol $b(h)$ given by

$$b(h) = (2\pi h)^{-n} \iint e^{-iy \cdot \zeta/h} C(x, x + y, \xi + \zeta)\, dy\, d\zeta, \tag{A.4}$$

where $C(x, y, \xi) = a_1(x, \eta(x, y, \xi))\ \overline{a_2(y, \eta(x, y, \xi))}\ J(x, y, \xi)$. Using the Taylor expansion for $C(x, x+y, \xi+\zeta)$ in ζ-variable, we get (A.3) with the remainder $r_N(h)$ given by

$$r_N(x, \xi; h) = (2\pi h)^{-n} \iint e^{-ih^{-1}y\cdot\eta} \int_0^1 (1-\sigma)^N \sum_{|\alpha| = N+1} \frac{1}{\alpha!} \partial_\xi^\alpha D_y^\alpha C(x, x+y, \xi+\sigma\eta)\, d\sigma\, dy\, d\eta. \tag{A.5}$$

To estimate $r_N(h)$, it suffices to integrate (A.5) for a sufficient number of times and to notice that for $0 < \theta < 1$, one has

$$|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma (\eta(x, x+y, \xi) - \xi)| \leqslant C_{\alpha\beta\gamma} \langle x\rangle^{-\varepsilon_0 - |\alpha| - |\beta|} \qquad \text{for } |y| \leqslant \theta|x|. \tag{A.6}$$

∎

By the expression (A.3), we see that b_k belongs to the class $S^{m_1+m_2-k}$. If $a \in S$ ($\equiv S^0$), Lemma A.1 shows that $I(a, h)\ I(a, h)^*$ is a h-pseudo-differential operator with bounded symbol. By the well-known Caldéron–Vaillancourt Theorem [4], $I(a, h)$ is uniformly continuous on $L^2(\mathbf{R}^n)$:

$$\sup_{h \in]0,1]} \|I(a, h)\|_{\mathscr{L}(L^2)} < +\infty. \tag{A.7}$$

In what follows we need a slightly generalized version of (A.7).

PROPOSITION A.2. *Let $m \in \mathbf{R}$ and $a \in S^m$. Then for every $s \in \mathbf{R}$, $I(a, h)$ defines a continuous operator from $L^{2,s+m}$ to $L^{2,s}$ and*

$$\sup_{h \in]0,1]} \|I(a, h)\|_{(s+m,s)} < +\infty.$$

Proof. It is sufficient to prove Proposition A.2 for $m = 0$ and $s > 0$. Without loss of generality we can assume that $\varepsilon_0 = 1/k$ for some $k \in \mathbf{N}^*$. The commutator $i[x_j, I(a, h)]$ is also a Fourier integral operator. More precisely for $a \in S$ we can write

$$i[x_j, I(a, h)] = \langle x\rangle^{1-\varepsilon_0} I(b_j(h), h), \qquad j = 1, 2, ..., n, \tag{A.8}$$

where $b_j(x, \xi; h) = \langle x\rangle^{\varepsilon_0 - 1}(i(x_j - \partial_{\xi_j}\phi(x, \xi))\, a(x, \xi) + h\, \partial_{\xi_j} a(x, \xi))$. By (3.1), $b_j(h)$ is in S. From (A.7) and (A.8) it follows that

$$\begin{aligned} \|\langle x\rangle^{\varepsilon_0} I(a, h) f\| &\leqslant C \left\{ \sum_{j=1}^n \|\langle x\rangle^{\varepsilon_0 - 1} I(a, h)\, x_j f\| + \|f\| \right\} \\ &\leqslant C\|\langle x\rangle f\| \qquad \text{if} \quad 0 < \varepsilon_0 \leqslant 1. \end{aligned} \tag{A.9}$$

Expression (A.9) is also true with a replaced by $b_j(h)$. Multiplying (A.8) by $\langle x \rangle^{2\varepsilon_0 - 1}$ on the left, we get

$$\|\langle x \rangle^{2\varepsilon_0} I(a, h) f\| \leqslant C \left\{ \sum_{j=1}^{n} \|\langle x \rangle^{\varepsilon_0} I(b_j, h) f\| + \|\langle x \rangle^{2\varepsilon_0 - 1} I(a, h)\, x_j f\| \right\}$$
$$\leqslant C \|\langle x \rangle f\|, \qquad \text{if} \quad 2e_0 \leqslant 1.$$

Repeating these arguments k times, using the fact $k\varepsilon_0 = 1$, we get

$$\|\langle x \rangle\, I(a, h) f\| \leqslant C \|\langle x \rangle f\|, \qquad f \in \mathscr{S}(\mathbf{R}^n).$$

This proves Proposition 3.2 for $m = 0$ and $s = 1$. For s an arbitrary integer, we can use the same commutator techniques to prove

$$\sup_{h \in]0,1]} \|I(a, h)\|_{\mathscr{L}(L^{2,s})} < +\infty.$$

For general $s \geqslant 0$, the result follows by interpolations. ▮

We give now a formula for the composition of Fourier integral operators with pseudo-differential operators.

PROPOSITION A.3. *Let $a_j \in S^{m_j}$ for $j = 1, 2$, Then $a_1(x, hD)\, I(a_2, h)$ is a Fourier integral operator with phase function ϕ,*

$$a_1(x, hD)\, I(a_2, h) = I(b(h), h),$$

where $b(h)$ admits an asymptotic expansion in h, for every $N \geqslant 0$,

$$b(x, \xi; h) = \sum_{j=0}^{N} h^j\, b_j(x, \xi) + h^{N+1} r_N(x, \xi; h) \tag{A.10}$$

with

$$b_0(x, \xi) = a_1(x, \nabla_x \phi(x, \xi))\, a_2(x, \xi)$$

$$b_l(x, \xi) = \sum_{l \leqslant |\alpha| \leqslant 2l} \frac{1}{\alpha!} \partial_\xi^\alpha a_1(x, \nabla_x \phi(x, \xi)) \sum_{\substack{\beta_1 + \cdots \beta_k + \alpha' = \alpha \\ k = |\alpha| - l \\ |\beta_1|, |\beta_2|, \ldots, |\beta_k| \geqslant 2}} C_{\beta_1 \cdots \beta_k}$$
$$\times D_x^{\alpha'} a_2(x, \xi)\, \partial_x^{\beta_1} \phi(x, \xi) \cdots \partial_x^{\beta_k} \phi(x, \xi)$$

and $\{r_N(h),\ h \in]0, 1]\}$ is bounded in $S^{m_1 + m_2 - N - 1}$:

$$|\partial_x^\alpha \partial_\xi^\beta r_N(x, \xi; h)| \leqslant C_{\alpha\beta} \langle x \rangle^{m_1 + m_2 - N - 1 - |\alpha|}, \qquad (x, \xi) \in \mathbf{R}^{2n}$$

uniformly in $h \in]0, 1]$.

Proof. Put $S(x, y, \eta) = \phi(y, \eta) - \phi(x, \eta) + (x-y) \cdot \nabla_x \phi(x, \eta)$. Then $b(h)$ is formally given by

$$b(x, \eta; h) = (2\pi h)^{-n} \iint e^{ih^{-1}(S(x, y, \eta) + (x-y)\cdot\xi)} a_1(x, \xi + \nabla_x \phi(x, \eta)) \times a_2(y, \eta)\, dy\, d\xi.$$

See Helffer–Robert [7]. By Taylor expansion for $a_1(x, \xi + \nabla_x \phi(x, \eta))$ at $\xi = 0$, we get (3.10) with $r_N(h)$ given by

$$r_N(x, \eta; h) = (2\pi h)^{-n} \iint e^{ih^{-1}(x-y)\cdot\xi} C_N(x, y; \xi, \eta; h)\, dy\, d\xi, \quad \text{(A.11)}$$

where

$$C_N(x, y; \xi, \eta; h) = \sum_{|\alpha| = N+1} \frac{1}{\alpha!} \int_0^1 (1-\sigma)^N \partial_\xi^\alpha a_1(x, \nabla_x \phi + \sigma\xi)\, D_y^\alpha (e^{ih^{-1}S(x, y, \eta)} a_2(y, \eta))\, d\sigma.$$

Take $\rho \subset C_0^\infty(\mathbf{R})$ so that $\rho(\tau) = 1$, if $\tau < 1$ and $\rho(\tau) = 0$, if $\tau > 2$. Set $\rho_1 = 1 - \rho$ and

$$C_{N,1}(h) = \rho(|x-y|^2/h)\, C_N(h)$$
$$C_{N,2}(h) = \rho_1(|x-y|^2/h)\, \rho(|x-y|/\varepsilon\langle x\rangle)\, C_N(h)$$
$$C_{N,3}(h) = \rho_1(|x-y|^2/h)\, \rho_1(|x-y|/\varepsilon\langle x\rangle)\, C_N(h).$$

Denote by $r_{N,j}(h)$ the remainder corresponding to $C_{N,j}(h)$ by the formula (A.11). On the support of $C_{N,1}(h)$, we have, for any $\alpha \in \mathbf{N}^n$,

$$|\partial_x^\alpha e^{ih^{-1}S(x, y, \eta)}| \leqslant C_\alpha h^{-|\alpha|/2} \langle x\rangle^{-|\alpha| - \varepsilon_0}.$$

Making use of the operator $1 - h^2 \Delta_y$ and integrating (A.11) for $r_{N,1}(h)$ by part for a sufficient number of times, we get

$$|r_{N,1}(x, \eta; h)| \leqslant C h^{-(N+n+1)/2} \langle x\rangle^{m_1 + m_2 - N - 1}, \qquad h \in]0, 1].$$

On the support of $C_{N,2}(h)$ and $C_{N,3}(h)$, we can use the relation

$$e^{ih^{-1}(x-y)\cdot\xi} = |x-y|^{-2}(-h^2(-h^2 \Delta_\xi e^{ih^{-1}(x-y)\cdot\xi}).$$

In particular for $\varepsilon > 0$ sufficiently small, we have

$$|\nabla_x \phi(y, \eta) - \nabla_x \phi(x, \eta)| \leqslant C|x-y| \langle x\rangle^{-1-\varepsilon_0} \qquad \text{on} \quad \operatorname{supp} C_{N,2}(h).$$

Hence for every $\alpha \in \mathbf{N}^n$,

$$|\partial_y^\alpha e^{ih^{-1}S(x,y,\eta)}| \leqslant C_\alpha h^{-|\alpha|}(1+|x-y|^{|\alpha|})\langle x\rangle^{-|\alpha|}, \qquad \text{on} \quad \operatorname{supp} C_{N,2}(h).$$

This enables us to prove, by a suitable integration by parts,

$$|r_{N,j}(x,\eta;h)| \leqslant Ch^N \langle x\rangle^{m_1+m_2-N-1}, \qquad j=2,3.$$

This proves

$$|r_N(x,\eta;h)| \leqslant Ch^{-(N+n+1)/2}\langle x\rangle^{m_1+m_2-N-1}. \tag{A.12}$$

It is clear from (A.10) that b_j belongs to $S^{m_1+m_2-j}$. Pushing farther the expansion for $b(h)$, we can write

$$r_N(h) = \sum_{j=N+1}^{2N+n+1} h^{j-N-1} b_j + h^{N+n+1} r_{2N+n+1}(h).$$

Consequently applying (A.12) with $2N+n+1$ instead of N, we get

$$|r_N(x,\eta;h)| \leqslant C_N \langle x\rangle^{m_1+m_2-N-1} \qquad (x,\eta)\in \mathbf{R}^{2n}$$

uniformly in $h \in]0,1]$. Similarly we can get the estimates over the derivatives of $r_N(h)$. The proposition is proved. ▌

For the composition of Fourier integral operators by pseudo-differential operators on the right, we have a similar result.

PROPOSITION A.4. *Let $a_j \in S^{m_j}$, $j=1,2$. Then $I(a_1,h)\,a_2(x,hD)$ is a Fourier integral operator, $I(a_1,h)\,a_2(x,hD) = I(b(h),h)$, where $b(h)$ has an asymptotic expansion of the form*

$$b(h) = \sum_{j=0}^{N} h^j b_j + h^{N+1} r_N(h)$$

with

$$b_0(x,\xi) = a_1(x,\xi)\, a_2(\nabla_\xi \phi(x,\xi),\xi)$$

$$b_l(x,\xi) = \sum_{l\leqslant|\alpha|\leqslant 2l} \left\{ \sum_{\substack{k=|\alpha|-l \\ \beta_1+\cdots+\beta_k+\alpha'=\alpha \\ |\beta_1|\geqslant 2,\ldots,|\beta_k|\geqslant 2}} C_{\alpha\beta}\, \partial_\xi^{\alpha'} a_1(x,\xi)\, \partial_\xi^{\beta_1}\phi(x,\xi)\cdots\partial_\xi^{\beta_k}\phi(x,\xi) \right\}$$
$$\times D_x^\alpha a_2(\nabla_\xi\phi(x,\xi),\xi)$$

and $\{r_N(h),\ 0<h\leqslant 1\}$ is bounded in $S^{m_1+m_2-N-1}$.

The proof of Proposition A.4 is the same as that of Proposition A.3 and we omit it.

ACKNOWLEDGMENTS

The author thanks D. Robert for helpful advice which improved this paper. He thanks also B. Helffer for constant encouragement and stimulating discussions related to this subject.

REFERENCES

1. S. AGMON, Some new results on spectral and scattering theory of differential operators on $\mathbf{R}^n$, Séminaire Goulaouic-Schwartz (1978/79), N° 2.
2. K. ASADA AND D. FUJIWARA, On some oscillatory integral transformation in $L^2(\mathbf{R}^n)$, *Japan. J. Math.* **4** (1978), 299–361.
3. R. BEALS, A general calculus of pseudo-differential operators, *Duke Math. J.* **42** (1975), 1–42.
4. A. CALDERON AND R. VAILLANCOURT, A class of bounded pseudo-differential operators, *Proc. Nat. Acad. Sci. USA* **69** (1972), 1185–1187.
5. J. CHAZARAIN, Spectre d'un hamiltonien quantique et mécanique classique, *Comm. Partial Differential Equations* **5** (1980), 595–644.
6. D. FUJIWARA, A construction of the fundamental solution for the Schrödinger equation, *J. Analyse Math.* **35** (1979), 41–96.
7. B. HELFFER AND D. ROBERT, Comportement semi-classique du spectre des hamiltoniens quantiques elliptiques, *Ann. Inst. Fourier (Grenoble)* **31** (1981), 169–233.
8. B. HELFFER AND D. ROBERT, Propriétés asymptotiques du spectre d'opérateurs pseudo-différentiels sur $\mathbf{R}^n$, Comm. *Partial Differential Equations* **7** (1982), 795–882.
9. B. HELFFER AND D. ROBERT, Calcul fonctionnel par la transformation de Melin et opérateurs admissibles, *J. Funct. Anal.* **53** (1983), 246–268.
10. L. HORMANDER, On the asymptotic distribution of eigenvalues of pseudo-differentials operators in $\mathbf{R}^n$, *Ark. Mat.* **17** (1979), 297–313.
11. H. ISOZAKI AND H. KITADA, Micro-local resolvent estimates for two body Schrödinger operators, *J. Funct. Anal.* **57** (1984), 270–300.
12. H. ISOZAKI AND K. KITADA, Modified wave operators with time-independent modifiers, *J. Fac. Sci. Univ. Tokyo* **32** (1985), 77–104.
13. A. JENSEN, Propagation estimates for Schrödinger-type operators, *Trans. Amer. Math. Soc.* **291** (1985), 129–144.
14. A. JENSEN, E. MOURRE, AND P. PERRY, Multiple commutator estimates and resolent smoothness in quantum scattering theory, *Ann. Inst. H. Poincaré* **41** (1984), 207–225.
15. H. KITADA, Time-decay of high energy part of the solution for a Schrödinger equation, *J. Fac. Sci. Univ. Tokyo* **31** (A) (1984), 109–146.
16. M. MURATA, High energy resolvent estimates. II. High order elliptic operators, *J. Math. Soc. Japan* **36** (1984), 1–10.
17. J. RAUCH, Local decay of scattering solutions to Schrödinger's equation, *Comm. Math. Phys.* **61** (1978), 149–168.
18. M. REED AND B. SIMON, "Methods of Modern Mathematical Physics IV," Academic Press, New York, 1978.
19. D. ROBERT AND H. TAMURA, Semi-classical bounds for resolvents of Schrödinger operators and asymptotics for scattering phases, Comm. *Partial Differential Equations* **9** (1984), 1017–1058.
20. D. ROBERT AND H. TAMURA, Semi-classical asymptotic for spectral function of Schrödinger operators and applications to scattering problems, to appear.

21. B. Vainberg, On the short wave asymptotic behavior of solutions of stationary problems and the asymptotic behavior as $t \to \infty$ of solutions of non-stationary problems, *Russian Math. Surveys* **30** (1975), 1–58.
22. X. P. Wang, Etude semi-classique d'observables quantiques, *Ann. Fac. Sci. Toulouse Math.* **7** (1985), 101–135.
23. X. P. Wang, Time-delay operators in semi-classical limit, finite-range potentials, *Ann. Scuola Norm. Sup. Pisa*, to appear.
24. X. P. Wang, Opérateurs de temps-retard dans la théorie de la diffusion, *C.R. Acad. Sci. Paris* **301** (1985), 789–891.
25. X. P. Wang, Approximation semi-classique de l'équation de Heisenberg, *Comm. Math. Phys.* **104** (1986), 77–86.
26. X. P. Wang, Décroissance en temps de la solution de l'équation de Schrödinger, exposé au séminaire. "Equations aux Dérivées Partielles," Nantes, Février 1986.
27. V. Enss, Propagation properties of quantum scattering states, *J. Funct. Anal.* **52** (1983), 219–251.
28. H. Isozaki, Differentiability of generalised Fourier transforms associated with Schrödinger operators, *J. Math. Kyoto Univ.* **25** (1985), 789–806.
29. H. Isozaki, Decay rates of scattering states for Schrödinger operators, preprint.